

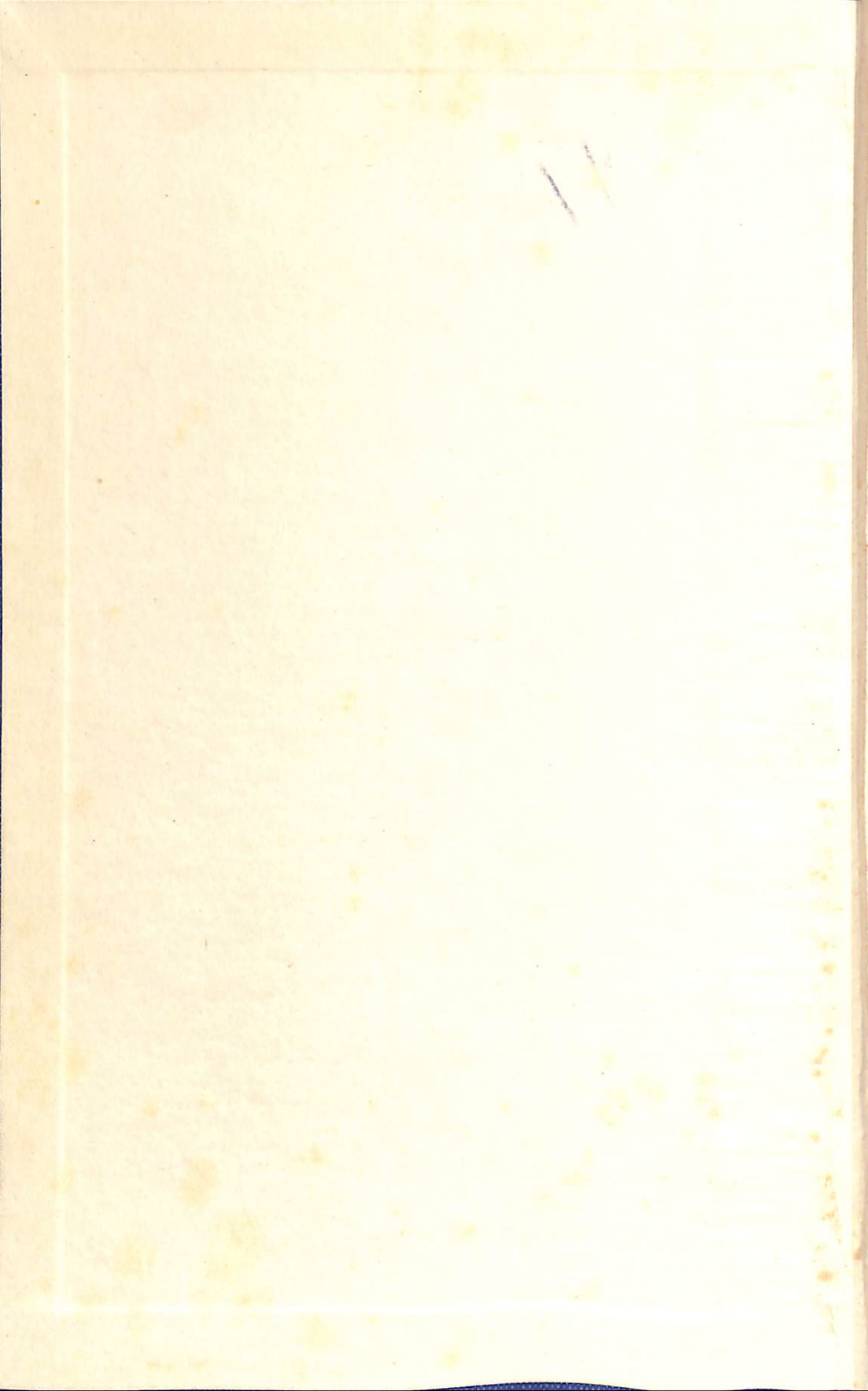
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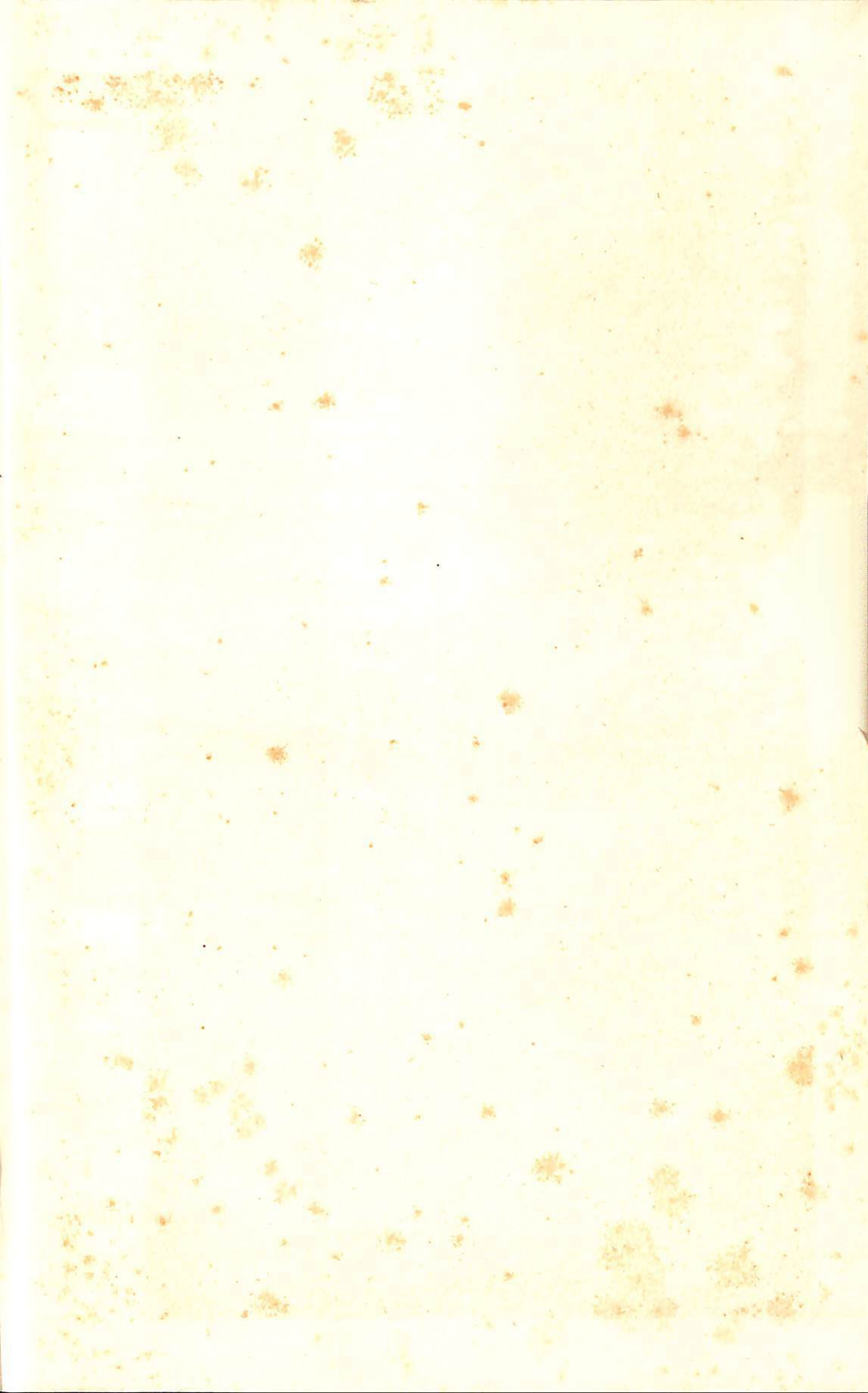
AERONAUTICAL ENGINEERING



VOLUME TWO

R. TATHAM





THE TECHNICAL COLLEGE SERIES

Editor Emeritus: P. ABBOTT, B.A.

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Principal, Wolverhampton and Staffordshire
Technical College

AERONAUTICAL ENGINEERING

VOLUME II

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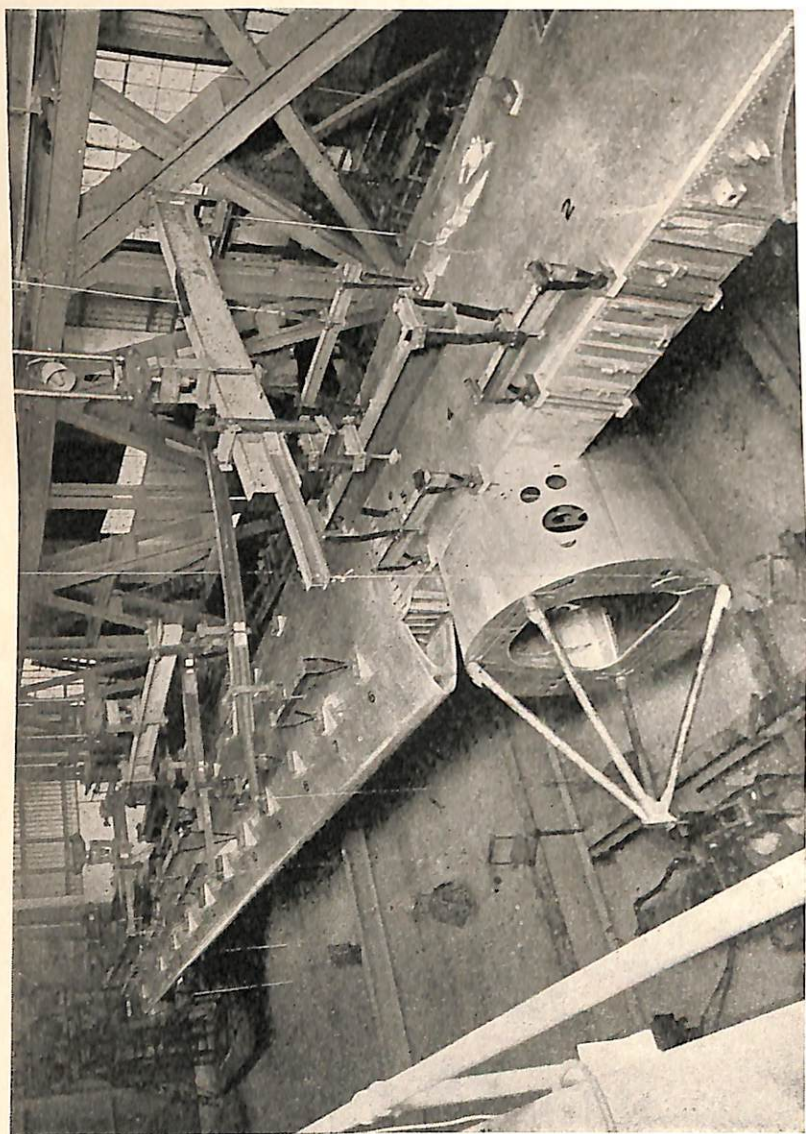
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STRUCTURAL TEST RIG FOR THE WING OF A LARGE CIVIL AEROPLANE.
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THE TECHNICAL COLLEGE SERIES

AERONAUTICAL ENGINEERING

FOR
NATIONAL CERTIFICATE

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VOLUME II
FUNDAMENTALS OF AIRCRAFT STRUCTURAL
ANALYSIS



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GENERAL EDITOR'S FOREWORD

THE Technical College Series to-day includes many books which are outstanding in their particular fields, and it is the aim of the publishers to maintain and develop the worthy tradition of the Series while meeting in full the increasing needs of technical and scientific education.

An outstanding contribution of the technical colleges to education has been the system of National Certificates under which the Ministry of Education and the Colleges work in association with leading professional institutions. The system has progressed from its early pre-occupation with engineering until the schemes now cover practically the whole field of higher technology and applied science. The major engineering institutions, the Royal Institute of Chemistry, the Institute of Physics, the Institution of Metallurgists are all associated with National Certificate schemes. There are National Certificates in Building and in Commerce, with each of which a group of professional institutions is associated. Though the pattern of National Certificate Courses was originally dictated by the needs and limitations of the evening student the system of endorsements obtainable by further study has now brought about the result that these courses have been extended to meet the full requirements of practice in the subjects with which they deal. During recent years the system of part-time day release of apprentices and learners has become common in all branches of industry as well as in the public services. This has effected something like a revolution in technical education ; and in particular the treatment of National Certificate studies up to the standard already indicated has become much broader.

The books included in the Series will be planned to suit the requirements of three main groups : (i) the part-time and full-time students working in technical colleges for professional qualifications and university degrees, (ii) technologists, managers, and research workers in industry, (iii) teachers in technical colleges and elsewhere who require text-books of high standard, but broad enough in treatment of their subjects to be readily adaptable to local approved schemes of study.

W. E. FISHER.

P R E F A C E

IN view of the difficulty in deciding what constitutes a reasonable year's work beyond the S3 stage, and in order to allow for differences in the order of treatment of the various portions of the work required for the higher courses, this volume does not continue all the subjects introduced in Volume I. Only the structural section of the first volume is carried further in this book, and it is hoped that the standard will prove sufficient for A1 and A2 courses for the Higher National Certificate in the fundamentals of the theory underlying the analysis and design of aircraft structures. Some emphasis has been placed on the basic principles of stressed skin structures, although it is considered to be neither possible nor desirable to attempt, in a book of this standard, a detailed exposition of the theory of such structures, so much of which is embodied in technical reports. It is intended rather to give the student some idea of the principles involved, even though it has been necessary to make simplifying assumptions regarding the effectiveness of the skin. Strain energy and the solution of redundant structures by use of the principle of least work appear at an earlier stage than is usual, primarily because the consideration of strain energy is such a fundamental method of approach to many problems, and it is not thought that the concept of partial differentiation, involving only first order differential coefficients, should prove to be very difficult to assimilate at this stage.

A number of examples to be worked by the student will be found at the end of the text. These examples and the worked examples embodied in the text have been drawn for the most part from papers set in the Associate Fellowship Examination of the Royal Aeronautical Society, and grateful acknowledgment is made of the permission, so kindly granted, to use these problems.

The photographs of the "Cathedral" test frame at the Royal Aircraft Establishment are published by permission of His Majesty's Stationery Office.

R. T.

RYDE.

REFERENCE TO VOLUME I

It is assumed throughout this volume that the student is familiar with the elementary principles covered in Volume I. Where reference to Volume I is necessary, such reference will be in the form of the appropriate paragraph or equation number preceded by the figure I. For example, I.8.3 means Volume I, Chapter 8, paragraph 3.

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CHAPTER I

SPACE FRAMES

1.1. In Volume I, Chapter VIII, methods were described for determining the loads in the members of a simply-stiff plane frame. Many frames used in aircraft structures are three-dimensional or space frames, and methods must be found for assessing the loads in the members of such frames. Although the frames may be redundant or simply-stiff, in this chapter we shall consider only simply-stiff frames. It may be remarked that the analysis of a redundant frame, whether in two or three dimensions, involves at some stage the analysis of a simply-stiff frame, so that we shall find the methods described here and in Volume I of use when we consider redundant frames later in this volume.

1.2. Relation between Number of Joints and Number of Members

By a method similar to that in para. I.8.3, we may derive a relationship between the number of joints and the number of members in a perfect space frame. The simplest space frame is a tetrahedron, (Fig. 1.1), which has four joints and six

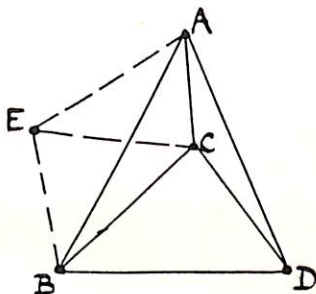


FIG. 1.1.—BUILD-UP OF PERFECT SPACE FRAME.

members, and is obtained by bracing one joint, A, in space to the simplest perfect plane frame, BCD. To brace any further joint, E, to this frame requires a minimum of three members. Consequently, if we have j joints altogether, we require six members for the first four joints and three members for each successive joint after the first four. Hence the total number of members, n , is given by

$$n = 6 + 3(j - 4) = 3j - 6 \quad \text{Eq. 1.1}$$

It may be noted here that this relationship could be obtained without reference to the basic tetrahedron, by remarking that each joint will yield three equations of equilibrium for the internal and external forces applied there. There will thus be $3j$ equations from which to determine the unknown external reactions and the internal loads. Of these $3j$ equations six will be required to find the external reactions, leaving $3j - 6$ equations to be solved for the internal loads. These serve to determine only $3j - 6$ loads so that for a statically-determinate frame we arrive at the relationship given in equation 1.1.

In the case of a frame braced to a rigid wall or bulkhead, a different relationship holds. The wall is assumed to provide sufficient members (in three dimensions) to brace together the number of joints on it. For example, if the total number of joints is j , and if of this number $\bar{j}$ are actually on the rigid wall, the

wall is assumed to provide $3j - 6$ members of the total required. Hence in this case the number, n_1 , of members required will be

$$n_1 = 3j - 6 - (3j - 6) = 3(j - J) \quad \text{Eq. 1.2}$$

For a two-dimensional frame the corresponding relationship is

$$n_1 = 2(j - J)$$

The relationship for a three-dimensional frame is illustrated by Fig. 1.2, where there are four joints altogether, three of which are on a rigid bulkhead. Applying equation 1.2 shows that the three members AD, BD, CD are sufficient to brace

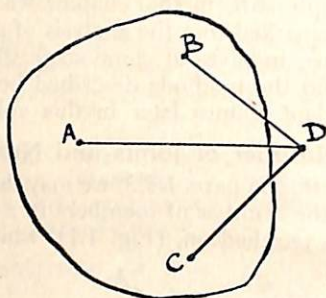


FIG. 1.2.—JOINT BRACED TO RIGID WALL.

the joint D to the wall. The wall itself provides the other three members necessary to give complete three-dimensional bracing between the four joints.

1.3. Determination of Loads in the Members of a Space Frame—Graphical Method

It is possible to determine the internal loads in the members of a simply-stiff space frame by a series of applications of the stress diagram for a plane frame (1.8.6). This requires that the space frame shall be split up into a number of plane frames, the applied loads resolved into these planes and stress diagrams drawn for each plane. Certain members may occur in more than one plane frame, and the loads obtained for such members from each plane frame in which they occur are finally added algebraically to give the resultant loads.

In general, the analytical method described in the next paragraph will be found to be more convenient than the graphical method given above.

1.4. Determination of Loads—Analytical Method

Probably the best-known and most convenient method of determining the loads in the members of a simply-stiff space frame is that known as the method of *Tension Coefficients*, due to Southwell. The tension coefficient of a member, OA, of a frame is defined as the tension per unit length, and is denoted by t_{OA} . Thus, for a member of length L , carrying a tensile load T , the tension coefficient is given by the equation

$$t = \frac{T}{L} \quad \text{Eq. 1.3}$$

If the force in a member is compressive, the tension coefficient for that member is negative. The use of tension coefficients in the analysis of a frame will be

illustrated first in general terms with reference to Fig. 1.3, and then by a worked example.

In the figure, OA represents a member of a frame, and x, y, z are the projected lengths of OA in directions parallel respectively to the three mutually perpendicular axes OX, OY, OZ . The length of OA is L . AP is perpendicular to the plane YOX , and PQ is perpendicular to OX . A force T along OA can be resolved into $T \sin \theta$ along OZ and $T \cos \theta$ along OP . The force $T \cos \theta$ along OP can be resolved further into $T \cos \theta \cos \phi$ along OX and $T \cos \theta \sin \phi$ along OY . We

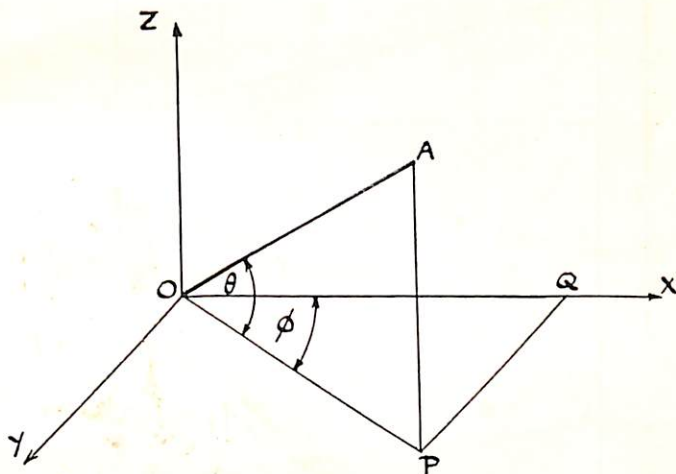


FIG. 1.3.—PROJECTED LENGTHS OF A LINE.

can express x, y, z in terms of L, θ, ϕ , by resolving the length L into its three components, as follows,

$$\begin{aligned}x &= L \cos \theta \cos \phi \\y &= L \cos \theta \sin \phi \\z &= L \sin \theta\end{aligned}$$

We can now express the components of T in terms of x, y, z .

$$\text{Component of } T \text{ in direction } OX = T \cos \theta \cos \phi = T \cdot \frac{x}{L} = t \cdot x$$

$$,, \quad ,, \quad T \quad ,, \quad OY = T \cos \theta \sin \phi = T \cdot \frac{y}{L} = t \cdot y$$

$$,, \quad ,, \quad T \quad ,, \quad OZ = T \sin \theta = T \cdot \frac{z}{L} = t \cdot z$$

Hence the component in any direction of the force in a member is the tension coefficient multiplied by the projected length of the member in that direction.

If now we have several members, $OA, OB, OC \dots$, meeting at O , and if the external forces at O have components X, Y, Z in the directions OX, OY, OZ respectively, we may write the equations of equilibrium for that joint as

$$\left. \begin{aligned}t_{OA}x_{OA} + t_{OB}x_{OB} + t_{OC}x_{OC} + \dots + X &= 0 \\t_{OA}y_{OA} + t_{OB}y_{OB} + t_{OC}y_{OC} + \dots + Y &= 0 \\t_{OA}z_{OA} + t_{OB}z_{OB} + t_{OC}z_{OC} + \dots + Z &= 0\end{aligned} \right\} \quad \text{Eqs. 1.4}$$

Three similar equations may be written for each joint, the origin being changed

each time to the joint considered, and the complete set of equations may be solved for the tension coefficients. The loads in the various members are then obtained by multiplying the tension coefficients by the lengths of the members. The main

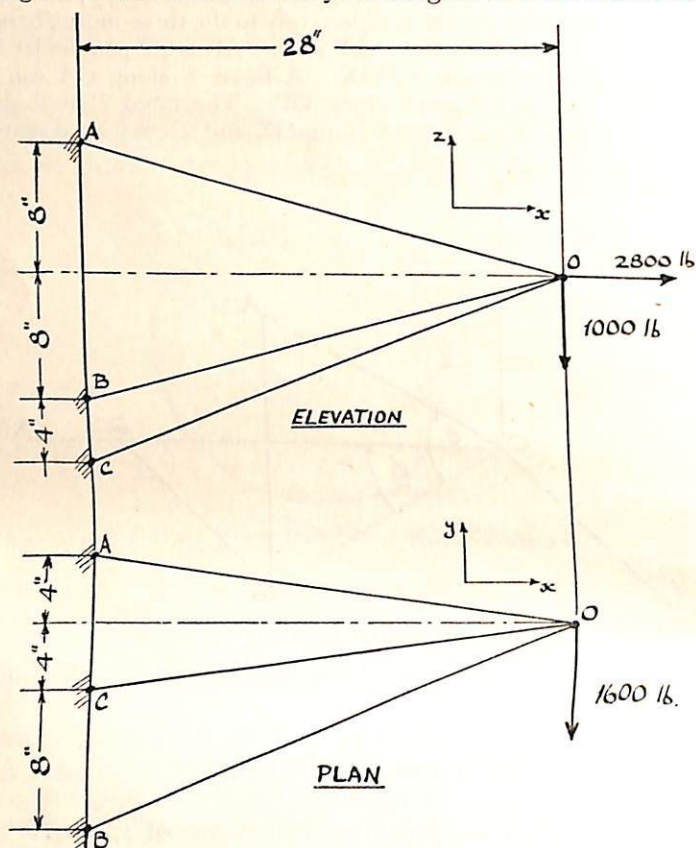


FIG. 1.4.

advantage of this method lies in the fact that x , y , z for any member can be determined directly from a dimensioned three-view drawing of the frame. The length, L , follows from $L = \sqrt{x^2 + y^2 + z^2}$.

Example 1.1. Determine the loads in the members of the frame shown in Fig. 1.4, using the method of tension coefficients.

Taking O as origin, let the positive directions of the x , y and z axes be as indicated in the figure. We write down the values of x , y , z , and L for each member as follows, all lengths being in inches:—

Member.	x	y	z	L
OA	- 28	4	8	29.4
OB	- 28	- 12	- 8	31.5
OC	- 28	- 4	- 12	30.7

Inserting these values in equations 1.4 we have

$$-28t_{OA} - 28t_{OB} - 28t_{OC} + 2800 = 0 \quad . \quad . \quad . \quad (1)$$

$$4t_{OA} - 12t_{OB} - 4t_{OC} - 1600 = 0 \quad . \quad . \quad . \quad (2)$$

$$8t_{OA} - 8t_{OB} - 12t_{OC} - 1000 = 0 \quad . \quad . \quad . \quad (3)$$

These three equations may, for convenience, be re-written as follows:—

$$(1) \text{ becomes } -t_{OA} - t_{OB} - t_{OC} + 100 = 0 \quad . \quad . \quad . \quad (4)$$

$$(2) \text{ becomes } t_{OA} - 3t_{OB} - t_{OC} - 400 = 0 \quad . \quad . \quad . \quad (5)$$

$$(3) \text{ becomes } t_{OA} - t_{OB} - \frac{3}{2}t_{OC} - 125 = 0 \quad . \quad . \quad . \quad (6)$$

$$\text{From (4) and (5)} \quad 4t_{OB} + 2t_{OC} + 300 = 0 \quad . \quad . \quad . \quad (7)$$

$$\text{From (4) and (6)} \quad -2t_{OB} - \frac{5}{2}t_{OC} - 25 = 0 \quad . \quad . \quad . \quad (8)$$

$$\text{From (7) and (8)} \quad -3t_{OC} + 250 = 0$$

$$t_{OC} = \frac{250}{3}$$

Then from (7)

$$t_{OB} = -\frac{350}{3}$$

and from (4)

$$t_{OA} = \frac{400}{3}$$

We have finally,

Member.	t	$T = tL$
OA	400/3	3920 lb. Tension
OB	- 350/3	3675 lb. Compression
OC	250/3	2558 lb. Tension

CHAPTER II

STRAIN ENERGY—THEOREMS AND APPLICATIONS

2.1. Introduction

IN aircraft structures many cases of redundant structures arise—structures which cannot be analysed by methods of statics alone. In such circumstances it generally, though not always, becomes necessary to investigate the internal *strain energy* of the structure. When a piece of material is strained within the elastic limit, energy is stored up in the material, representing the capacity of the piece to return to its original dimensions on removal of the applied load. Provided the elastic limit is not exceeded, the internal strain energy will be equal to the work done by the applied forces and moments. If the elastic limit is exceeded, then some of the work done by the external forces is used in overcoming frictional resistance between the molecules of the material and is converted into heat energy instead of being available as strain energy to restore the material to its original dimensions. In this chapter, we shall consider some fundamental theorems relating to strain energy, and we shall give some applications to problems which, in certain cases, will be treated later by other methods. First, we must see how the internal strain energy can be expressed in terms of the applied loading.

2.2. Strain Energy in Tension or Compression

If a bar of length L and cross-sectional area A elongates an amount δL when subjected to a tensile load P , as in Fig. 2.1, the load-extension diagram for a

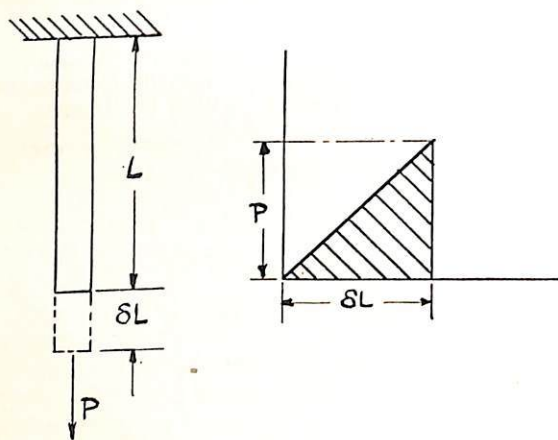


FIG. 2.1.—STRAIN ENERGY IN TENSION.

gradually applied load will be as shown, and the work done by the external force will be equal to the area of the shaded portion of the diagram, which is $\frac{1}{2}P\delta L$. Since the strain energy must be equal to the external work done, the strain energy, U , is given by

$$U = \frac{1}{2} \cdot P\delta L$$

But
$$\delta L = L \times \text{strain} = L \times \frac{\text{stress}}{E} = \frac{L}{E} \cdot \frac{P}{A}$$

Therefore
$$U = \frac{P}{2} \cdot \frac{PL}{AE} = \frac{P^2 L}{2AE} \quad \text{Eq. 2.1}$$

Alternatively, if p is the tensile stress, equal to P/A , we may write

$$U = \frac{p^2}{2E} \cdot AL = \frac{p^2}{2E} \times (\text{volume of the bar})$$

The strain energy per unit volume is therefore $\frac{p^2}{2E}$, and is known as the *resilience*.

2.3. Strain Energy in Shear

For a block ABCD, cross-sectional area A and length L , subjected to a shear force F , with shear strain ϕ , as indicated in Fig. 2.2, we find

$$\begin{aligned} \text{strain energy} = U &= \text{work done by } F \text{ in moving through distance } BB' \\ &= \frac{F}{2} \cdot BB' = \frac{F}{2} \cdot L \tan \phi = \frac{F}{2} \cdot L \phi \text{ for small values of } \phi \end{aligned}$$

But $\phi = \text{shear strain} = \frac{\text{shear stress}}{G} = \frac{F}{AG}$, so that

$$U = \frac{F^2 L}{2AG} = \frac{q^2 AL}{2G}, \text{ where } q = \text{shear stress} \quad \text{Eq. 2.2}$$

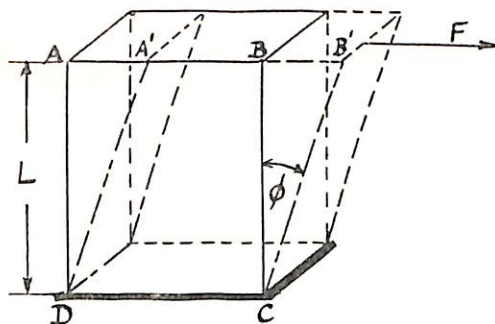


FIG. 2.2.—STRAIN ENERGY IN SHEAR.

2.4. Strain Energy in Bending

Consider an elementary length δx of a beam, which is initially straight, and assume that under the action of a bending moment M the element bends as shown in Fig. 2.3 (a), the radius of curvature of the neutral surface being R . If we increase the bending moment gradually from zero to M , a diagram showing the rotation, θ , plotted against the moment will be a straight line, as in Fig. 2.3 (b). The work done by the externally applied forces is therefore the shaded area of this diagram, which is equal to $\frac{M\theta}{2}$. Hence, the strain energy of this element of the beam is given by

$$\delta U = \frac{M\theta}{2}$$

But, $\theta = \frac{\delta x}{R} = \delta x \cdot \frac{M}{EI}$, since, from I.11.5, $\frac{1}{R} = \frac{M}{EI}$

We now have $\delta U = \frac{M^2 \delta x}{2EI}$

and for the whole beam, of length L , the strain energy, U , is

$$U = \int_0^L \frac{M^2 dx}{2EI} \quad \dots \quad \text{Eq. 2.3}$$

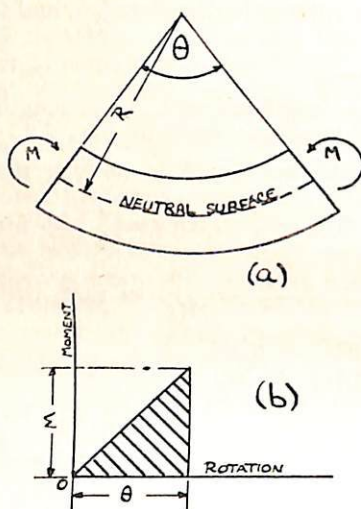


FIG. 2.3.—STRAIN ENERGY IN BENDING.

2.5. Strain Energy in Torsion

If a bar of length L twists through an angle θ when subjected to a gradually applied torque T , the work done is $\frac{T\theta}{2}$. From equation I.12.5, for a circular section bar,

$$\theta = \frac{TL}{GJ}$$

(A similar relation holds for sections other than circular, J being the torsion constant, which is equal to the polar second moment of area for a circular section.)

Since the strain energy is equal to the external work done, it follows that

$$U = \frac{T\theta}{2} = \frac{T^2 L}{2GJ} \quad \dots \quad \text{Eq. 2.4}$$

The above result assumes T constant over the whole length. If T varies, we must consider an elementary length, as in para. 2.4, and integrate over the length of the bar. Thus, we obtain as a general expression,

$$U = \int_0^L \frac{T^2 dx}{2GJ} \quad \dots \quad \text{Eq. 2.5}$$

It should be noted that, in the derivation of all the above expressions for strain energy, it has been assumed that the load, bending moment or torque is applied gradually. If the application of the load or moment is instantaneous, vibrations will be set up. These vibrations, however, will be damped, and, provided the elastic limit is not exceeded at any time, the final strain energy will be the same as if the load were applied gradually, although instantaneously the strain energy will exceed this value. The effects of suddenly applied loads will be considered briefly in para. 2.12.

2.6. Applications

Before proceeding to some fundamental theorems relating to strain energy, we will consider some simple applications of the above results.

(i) DEFLECTION OF A LOADED POINT IN A PIN-JOINTED FRAME

Example 2.1. The frame shown in Fig. 2.4 is pinned to a rigid wall at A and E, and all the joints are pin joints. The cross-sectional area of each member is 0.5 sq. in., and $E = 10^7$ lb. per sq. in.

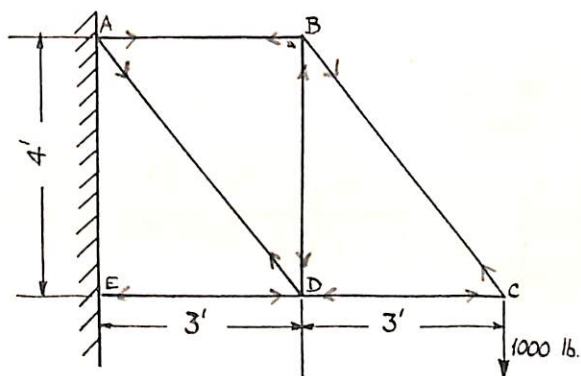


FIG. 2.4.—EXAMPLE 2.1.

The internal strain energy of the complete frame will be equal to the sum of the strain energy of all the individual members. Hence,

$$U = \sum \frac{P^2 L}{2AE} \quad \text{Eq. 2.6}$$

where P represents the load in any member of length L and cross-sectional area A . The value of P for each member may be determined by any of the methods outlined in Volume I, Chapter VIII, and U may then be evaluated from equation 2.6. Since the internal strain energy is equal to the external work done, we may write

$$U = \frac{1000}{2} \cdot \delta$$

where δ is the deflection of the point C in the direction of the load of 1000 lb. applied there. (Note that the external reactions at A and E do no work, since

the points at which they are applied, being attached to a rigid wall, do not move.)

The table below shows the calculations :—

Member.	L in.	A sq. in.	P lb.	$\frac{P^2 L}{2AE}$
AB	36	0.5	750	2.025
BC	60	0.5	1250	9.375
CD	36	0.5	— 750	2.025
BD	48	0.5	— 1000	4.80
DE	36	0.5	— 1500	8.10
DA	60	0.5	1250	9.375
				35.700

$$\therefore \frac{1000 \cdot \delta}{2} = U = 35.7$$

and

$$\delta = 0.0714 \text{ in.}$$

(ii) DEFLECTION OF A CANTILEVER BEAM

Example 2.2. Fig. 2.5 represents a cantilever beam, of constant cross-section, with second moment of area I , carrying a concentrated load W at the free end.

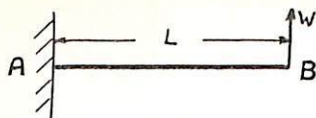


FIG. 2.5.—EXAMPLE 2.2.

At a distance x from the tip, the bending moment M is equal to Wx . Hence using equation 2.3, we obtain the strain energy

$$U = \int_0^L \frac{M^2 dx}{2EI} = \frac{1}{2EI} \int_0^L M^2 dx$$

since E and I are constant. Inserting the value of M and performing the integration, we have

$$U = \frac{W^2 L^3}{6EI}$$

Then, if the deflection of the tip, B, is y_B , equating the external work done to the strain energy gives

$$\frac{W}{2} y_B = \frac{W^2 L^3}{6EI}$$

i.e.,

$$y_B = \frac{WL^3}{3EI} \quad \text{Eq. 2.7}$$

This result will be seen to agree with that obtained by another method in Chapter VIII.

By the principle of superposition, no matter in what order we apply P_A and P_B , the final displacements, and hence the external work done and the internal strain energy, will be the same. Let us suppose that P_A is first applied alone. Then the work done at this stage is $\frac{1}{2}P_A(P_A \cdot \Delta_{aa})$. Now suppose P_B is added. This produces displacements of $P_B \cdot \Delta_{ab}$ at A and $P_B \cdot \Delta_{bb}$ at B. The work done due to the addition of P_B will be therefore $\frac{1}{2}P_B(P_B \cdot \Delta_{bb}) + P_A(P_B \cdot \Delta_{ab})$. The total work done is obtained by adding together the expressions for the work done in the two loading stages, and we have,

internal strain energy $U = \text{external work done}$

$$\text{i.e.} \quad U = \frac{1}{2}P_A(P_A \cdot \Delta_{aa}) + \frac{1}{2}P_B(P_B \cdot \Delta_{bb}) + P_A(P_B \cdot \Delta_{ab})$$

These expressions for the work done in the two loading stages may be obtained quite easily by reference to the load-displacement diagrams shown in Fig. 2.7, the work done being equal to the areas of the diagrams.

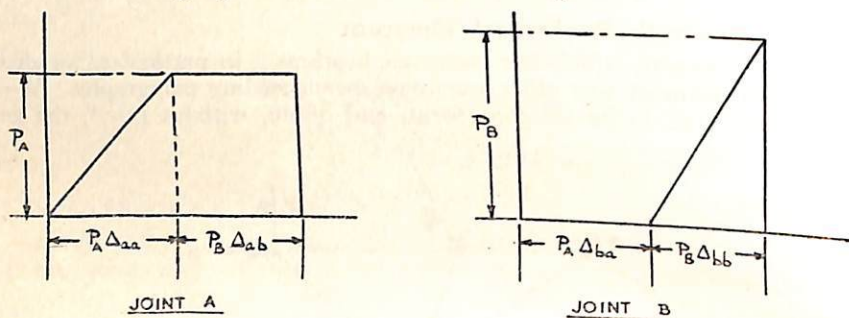


FIG. 2.7.—LOAD DISPLACEMENT DIAGRAMS FOR JOINTS A AND B OF FRAME OF FIG. 2.6, WHEN LOAD P_A IS APPLIED FIRST, FOLLOWED BY APPLICATION OF LOAD P_B .

Similarly, if we apply P_B first, and then add P_A , we obtain

$$U = \frac{1}{2}P_B(P_B \cdot \Delta_{bb}) + \frac{1}{2}P_A(P_A \cdot \Delta_{aa}) + P_B(P_A \cdot \Delta_{ba})$$

Since the order of application of the loads does not affect the final value of the strain energy, we can equate the two expressions obtained above for U , and we have finally,

$$\Delta_{ab} = \Delta_{ba}$$

i.e., *The displacement of A in the direction of P_A due to unit load at B is equal to the displacement of B in the direction of P_B due to unit load at A.*

The above is Clerk-Maxwell's reciprocal theorem in its simplest form. It can be extended, and in its most general form may be stated as follows:—If a structure is acted on by a system of forces $P_1, P_2, \dots P_n$ and couples $M_1, M_2, \dots M_n$, and if $\Delta_1, \Delta_2, \dots \Delta_n$ represent the displacements of the points of application of $P_1, P_2, \dots P_n$ respectively in the directions of those forces, whilst $\theta_1, \theta_2, \dots \theta_n$ represent the angular movements of the points of application of the couples $M_1, M_2, \dots M_n$ respectively, then the replacing of this system by another system $P'_1, P'_2, \dots P'_n, M'_1, M'_2, \dots M'_n$ with corresponding displacements and rotations $\Delta'_1, \Delta'_2, \dots \Delta'_n, \theta'_1, \theta'_2, \dots \theta'_n$ will result in the relationship

$$P_1 \Delta'_1 + P_2 \Delta'_2 + \dots + P_n \Delta'_n + M_1 \theta'_1 + \dots + M_n \theta'_n = P'_1 \Delta_1 + \dots + P'_n \Delta_n + M'_1 \theta_1 + \dots + M'_n \theta_n$$

or

$$\Sigma P \Delta' + \Sigma M \theta' = \Sigma P' \Delta + \Sigma M' \theta$$

It may be verified that this reduces to the simple form already obtained if we write

$$\begin{aligned} P_1 &= P_A; \quad P_2 = 0; \quad \Delta_1 = P_A \cdot \Delta_{aa}; \quad \Delta_2 = P_A \cdot \Delta_{ba} \\ P'_1 &= 0; \quad P'_2 = P_B; \quad \Delta'_1 = P_B \cdot \Delta_{ab}; \quad \Delta'_2 = P_B \cdot \Delta_{bb} \end{aligned}$$

Applications. This theorem is capable of many useful applications. We shall see it applied in the development of further strain energy theorems in succeeding paragraphs, and some other simple examples of its use may be mentioned here.

For instance, the deflection at the tip of a cantilever beam due to the application of a load anywhere along the beam may be determined simply by evaluating the deflection at the desired loading point due to a load of the same magnitude applied at the tip. It is an easy matter to obtain the deflection curve of a cantilever loaded at the tip, as will be seen in Chapter VIII, and the ordinate of such a curve at any point X gives the deflection at the tip when the load is applied at X. This example may be adapted also for experimental use. If it is desired to determine experimentally the deflection curve of a cantilever loaded at the tip, it is necessary only to take deflection measurements at the tip, the load being applied at a series of stations along the beam. Application of the reciprocal theorem shows that the tip deflection corresponding to the application of the load at a point X is the same as the deflection at the point X when the load is applied at the tip.

The slope at any point of a beam may also be determined by use of this theorem, since the slope at a point A when a unit load is applied at B will be of the same magnitude as the deflection at B when a unit couple is applied at A. It may be possible often to reduce the amount of work required to analyse a given structure by application of this theorem. It lends itself very well to experimental use.

2.9. Castigliano's First Theorem

This theorem may be stated as follows:—If the strain energy of any elastic structure is expressed in terms of the external loads, the *partial differential coefficient* * of the strain energy with respect to any one of the external loads is equal to the deflection of the point of application of that load in the direction of the load.

There are several ways of proving this theorem, some of which are simple fundamentally, but which are somewhat involved in their development and do not always indicate the general nature of the theorem. All the proofs involve the application of the reciprocal theorem, and the following proof has been adopted for presentation here because it is both simple and general.

Let $W_1, W_2, \dots, W_n$ represent a system of external loads applied to an elastic structure, producing deflections at the points of application, in the directions of the loads, of magnitudes $y_1, y_2, \dots, y_n$ respectively.

* If a quantity, U , is a function of several independent variables $W_1, W_2, W_3, \dots, W_n$, the partial differential coefficient of U with respect to any one of those variables is obtained by differentiating with respect to that variable in the usual way, assuming all the other independent variables to remain *constant*. e.g., if $U = aW_1 + bW_2 + \dots + nW_n$, where $a, b, \dots, n$ are constants, the partial differential coefficient of U with respect to W_n is

$$\frac{\partial U}{\partial W_n} = n$$

$W_1, W_2, \dots, W_{n-1}$ are treated as constants for the purpose of this differentiation. The partial differential coefficient is distinguished from the ordinary total differential coefficient by the use of the symbol ∂ instead of d .

Then the strain energy, U , is given by

$$U = \frac{1}{2}W_1y_1 + \frac{1}{2}W_2y_2 + \dots + \frac{1}{2}W_ny_n$$

Now let each load receive a small increment, so that the loads become $(W_1 + \delta W_1)$, $(W_2 + \delta W_2)$, $\dots$ $(W_n + \delta W_n)$. The corresponding deflections will be $(y_1 + \delta y_1)$, $(y_2 + \delta y_2)$, $\dots$ $(y_n + \delta y_n)$, and the strain energy will now be $(U + \delta U)$, given by

$$(U + \delta U) = \frac{1}{2}(W_1 + \delta W_1)(y_1 + \delta y_1) + \dots + \frac{1}{2}(W_n + \delta W_n)(y_n + \delta y_n)$$

Neglecting terms which are products of small quantities (such as $\frac{1}{2}\delta W_1\delta y_1$, etc.) we obtain by subtraction,

$$\delta U = \frac{1}{2}(W_1\delta y_1 + W_2\delta y_2 + \dots + W_n\delta y_n) + \frac{1}{2}(\delta W_1y_1 + \delta W_2y_2 + \dots + \delta W_ny_n)$$

Now by Clerk-Maxwell's reciprocal theorem we may write

$$W_1\delta y_1 = \delta W_1y_1; \quad W_2\delta y_2 = \delta W_2y_2; \quad \text{etc.},$$

so that

$$\delta U = \delta W_1y_1 + \delta W_2y_2 + \dots + \delta W_ny_n$$

$$\text{i.e.,} \quad \frac{\delta U}{\delta W_1} = y_1 + \frac{\delta W_2}{\delta W_1}y_2 + \dots + \frac{\delta W_n}{\delta W_1}y_n$$

If now we assume that only W_1 receives a small increment, whilst $W_2, \dots, W_n$ all remain constant, then $\delta W_2 = \delta W_3 = \dots = \delta W_n = 0$, and $\frac{\delta U}{\delta W_1}$ becomes the partial differential coefficient of U with respect to W_1 , and we have

$$\frac{\partial U}{\partial W_1} = y_1$$

which proves the theorem as stated above.

For a framework, in which the loads in the members are all axial loads, we know from equation 2.6 that the strain energy may be written as $U = \Sigma \frac{P^2L}{2AE}$, where P is the load in a member of length L and cross-sectional area A . The deflection, y_n , at the point of application of a load, W_n , in the direction of W_n , is, by the theorem proved above,

$$\begin{aligned} y_n &= \frac{\partial U}{\partial W_n} = \frac{\partial}{\partial W_n} \left(\Sigma \frac{P^2L}{2AE} \right) \\ &= \Sigma \frac{\partial}{\partial W_n} \left(\frac{P^2L}{2AE} \right) \\ &= \Sigma \frac{PL}{AE} \frac{\partial P}{\partial W_n} \dots \dots \dots \text{Eq. 2.8} \end{aligned}$$

This last form is very useful if there are a large number of loading points at which the deflections are to be determined. Numerical examples follow to illustrate this method of evaluating deflections of frames.

Example 2.3. Determine the deflection of the point C in the frame of Fig. 2.4, using the method derived above.

We must first determine the strain energy of the frame in terms of an arbitrary load, W , applied at C, in the direction in which we wish to know the deflection—i.e., we apply a load W at C instead of a load of 1000 lb. as shown in Fig. 2.4. We then calculate the quantities in the following table:—

Member.	L in.	A sq. in.	P lb.	$\frac{P^2 L}{2AE}$
AB	36	0.5	$0.75W$	$\frac{9}{16} \cdot \frac{W^2 \times 36}{1 \times 10^7} = \frac{81W^2}{4 \times 10^7}$
BC	60	0.5	$1.25W$	$\frac{25}{16} \cdot \frac{W^2 \times 60}{10^7} = \frac{375W^2}{4 \times 10^7}$
CD	36	0.5	$-0.75W$	$\frac{9}{16} \cdot \frac{W^2 \times 36}{10^7} = \frac{81W^2}{4 \times 10^7}$
BD	48	0.5	$-1.00W$	$\frac{W^2 \times 48}{10^7} = \frac{192W^2}{4 \times 10^7}$
DE	36	0.5	$-1.50W$	$\frac{9}{4} \cdot \frac{W^2 \times 36}{10^7} = \frac{324W^2}{4 \times 10^7}$
DA	60	0.5	$1.25W$	$\frac{25}{16} \cdot \frac{W^2 \times 60}{10^7} = \frac{375W^2}{4 \times 10^7}$
				By addition, $U = \frac{1428W^2}{4 \times 10^7}$

$$y_W = \frac{\partial U}{\partial W} = \frac{2 \times 1428W}{4 \times 10^7} = \frac{714W}{10^7}$$

Hence, for a load of 1000 lb. at C, the deflection is given by

$$y_C = \frac{714 \times 1000}{10^7} = 0.0714 \text{ in.}$$

which agrees with the result obtained previously in Example 2.1.

The advantages of the method become more apparent when there are several

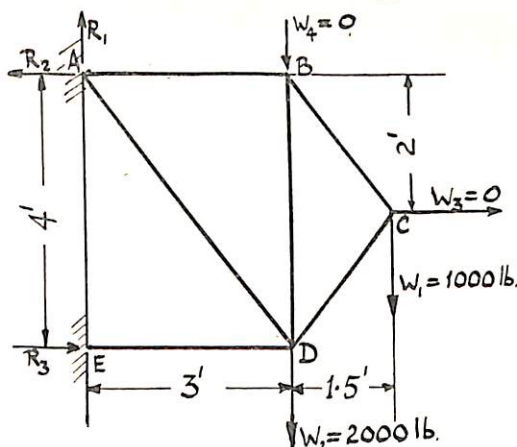


FIG. 2.8.—EXAMPLE 2.4.

loading points, with the deflection required at each, and when it is desired to determine the deflection of an unloaded point or the deflection of a loaded point in a direction other than that of the applied load. The following example will illustrate the general method:—

Example 2.4. The pin-jointed frame shown in Fig. 2.8 is pinned to a rigid

wall at A and E. The cross-sectional area of all the members is 1.0 sq. in., and $E = 10^7$ lb. per sq. in. For loads of 1000 lb. at C and 2000 lb. at D, both applied vertically downwards, determine

- (i) the horizontal and vertical deflections of C
- (ii) the vertical deflection of D
- (iii) the vertical deflection of B.

It will be clear from the previous example that we must determine the loads in the members, and hence the strain energy, in terms of general loads W_1 and W_2 instead of using the actual values of 1000 lb. and 2000 lb. respectively. Further, in order to obtain the horizontal deflection of C and the vertical deflection of B, we introduce fictitious loads W_3 and W_4 in the directions of the required deflections, as shown in the figure. When we have performed the necessary partial differentiations, the deflections for the given loading will be obtained by inserting the correct values of the loads. The loads in the members having been determined, in terms of W_1 , W_2 , W_3 and W_4 , by any of the methods given in Volume I, Chapter VIII, the following table may be set out. (In this table, a convenient way of expressing the loads is employed, the four columns under P giving the coefficients of the W 's for the load in the member concerned. For example, the load in AB is $0.375W_1 + 0.5W_3$, whilst that in DE is

$$-1.125W_1 - 0.75W_2 + 0.5W_3 - 0.75W_4.)$$

Member.	L in.	A sq. in.	P				$\frac{PL}{AE}$ (actual).
			W_1	W_2	W_3	W_4	
AB . .	36	1.0	0.375	0.00	0.500	0.000	13.50×10^{-4}
BC . .	30	1.0	0.625	0.00	0.833	0.000	18.75×10^{-4}
CD . .	30	1.0	-0.625	0.00	0.833	0.000	-18.75×10^{-4}
DE . .	36	1.0	-1.125	-0.75	0.500	-0.750	-94.50×10^{-4}
BD . .	48	1.0	-0.500	0.00	-0.667	-1.000	-24.00×10^{-4}
DA . .	60	1.0	1.250	1.25	0.000	1.250	225.00×10^{-4}

It will be observed that in the above table, the figures in the column headed W_1 are the values of $\frac{\partial P}{\partial W_1}$, those under W_2 are the values of $\frac{\partial P}{\partial W_2}$, and so on. The final column gives the value for each member of the quantity $\frac{PL}{AE}$, with the actual values of the W 's.

The required deflections are now evaluated as follows:—

$$\begin{aligned}
 y_1 &= \frac{\partial U}{\partial W_1} \text{ with the } W\text{'s given their correct values} \\
 &= \Sigma \left(\frac{PL}{AE} \cdot \frac{\partial P}{\partial W_1} \right) \text{ with the } W\text{'s given their correct values} \\
 &= \Sigma (\text{figure in col. 4 multiplied by figure in col. 8}) \\
 &= 10^{-4} (0.375 \times 13.5 + 0.625 \times 18.75 + 0.625 \times 18.75 \\
 &\quad + 1.125 \times 94.5 + 0.5 \times 24 + 1.25 \times 225) \\
 &= 10^{-4} (5.06 + 11.72 + 11.72 + 106.3 + 12.0 + 281.25) \\
 &= 0.0428 \text{ in.}
 \end{aligned}$$

Similarly,

$$\begin{aligned} y_2 &= \Sigma (\text{figure in col. 5 multiplied by figure in col. 8}) \\ &= 10^{-4}(0.00 + 0.00 + 0.00 + 70.875 + 0.00 + 281.25) \\ &= 0.0352 \text{ in.} \end{aligned}$$

$$\begin{aligned} y_3 &= \Sigma (\text{figure in col. 6 multiplied by figure in col. 8}) \\ &= 10^{-4}(6.75 + 15.62 - 15.62 - 47.25 + 16.0 + 0.0) \\ &= -0.0025 \text{ in.} \quad (\text{The negative sign indicates that the deflection is in the} \\ &\quad \text{direction opposite to that assumed for } W_3.) \end{aligned}$$

$$\begin{aligned} y_4 &= \Sigma (\text{figure in col. 7 multiplied by figure in col. 8}) \\ &= 10^{-4}(0.00 + 0.00 + 0.00 + 70.875 + 24.00 + 281.25) \\ &= 0.0376 \text{ in.} \end{aligned}$$

The whole of the above calculations could have been included in the table, by the addition of columns showing the products of each of cols. 4, 5, 6 and 7 with col. 8. The sum of the figures in each of the columns so obtained would then give the required deflections.

It should be noted that the theorem of this paragraph may be applied also

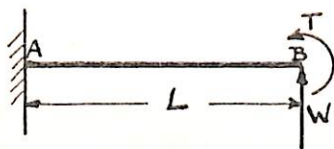


FIG. 2.9.—SLOPE AND DEFLECTION AT TIP OF A CANTILEVER.

to moments and the angular movements associated with them, as well as to forces and linear deflections. As an example of this application, consider the slope of a cantilever AB at the tip, B, due to a load W and a couple T , applied as shown in Fig. 2.9. At distance x from B the bending moment, M , is $Wx + T$. The strain energy, from equation 2.3, is

$$U = \int_0^L \frac{M^2 dx}{2EI} = \frac{1}{2EI} \int_0^L (Wx + T)^2 dx$$

If the couple T turns through an angle θ , then, with the assumption that plane cross-sections remain plane, θ will give the slope of the beam at B, and, applying Castiglano's first theorem, we have

$$\theta = \frac{\partial U}{\partial T} = \frac{1}{2EI} \int_0^L \frac{\partial (Wx + T)^2}{\partial T} \cdot dx = \frac{1}{2EI} \int_0^L 2(Wx + T) dx$$

$$\text{i.e.,} \quad \theta = \frac{WL^2 + 2TL}{2EI}$$

If in this expression we put $T = 0$, we obtain the slope at the tip of a cantilever, with a single load W at the tip, as $\frac{WL^2}{2EI}$. This result will be verified in Chapter VIII

by other means. We may note that $\frac{\partial U}{\partial W}$ gives the tip deflection, which will be

seen to be $\frac{WL^3}{3EI}$, when T is zero, agreeing with equation 2.7.

2.10. Castigliano's Second Theorem and the Principle of Least Work

The theorem proved in para. 2.9 may be applied to a redundant structure to yield Castigliano's second theorem, of which the principle of least work, or minimum strain energy, may be regarded as a special case.

In Fig. 2.10, let M and N represent two adjacent joints in a framed structure,

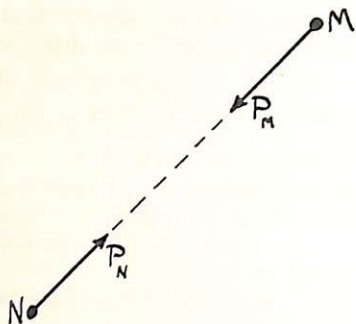


FIG. 2.10.—RELATIVE MOVEMENT OF TWO ADJACENT JOINTS IN A FRAME.

and let forces P_M and P_N be applied at these joints in the line MN as indicated. Suppose that the strain energy of the frame is U' . By Castigliano's first theorem, we may write

$$\frac{\partial U'}{\partial P_M} + \frac{\partial U'}{\partial P_N} = \text{amount by which distance MN is shortened,}$$

since $\frac{\partial U'}{\partial P_M} = \text{movement of joint M in direction MN}$

and $\frac{\partial U'}{\partial P_N} = \text{ " " " N " " NM}$

If P_M and P_N , and hence U' , depend on another quantity R , it can be shown that

$$\frac{\partial U'}{\partial R} = \frac{\partial U'}{\partial P_M} \cdot \frac{\partial P_M}{\partial R} + \frac{\partial U'}{\partial P_N} \cdot \frac{\partial P_N}{\partial R}$$

Now suppose that M and N are connected by a member, in which the load is R . Then $P_M = P_N = R$, and $\frac{\partial P_M}{\partial R} = \frac{\partial P_N}{\partial R} = 1$.

Hence, $\frac{\partial U'}{\partial R} = \frac{\partial U'}{\partial P_M} + \frac{\partial U'}{\partial P_N} = \text{shortening of distance MN}$

Then if L is the initial distance between M and N, the final distance between these two joints is $\left(L - \frac{\partial U'}{\partial R}\right)$.

Suppose that the initial length of the member MN is $(L - \lambda)$, so that it is too short to fit exactly between M and N, and let its cross-sectional area be A .

The load in the member being R , the stress is R/A , the strain is R/AE , and the extension is $\frac{R}{AE}(L - \lambda)$. The final length of the member is therefore

$$(L - \lambda) + \frac{R}{AE}(L - \lambda) = (L - \lambda)\left(1 + \frac{R}{AE}\right)$$

This must be equal to the final distance between the two joints, M and N, so that we have

$$\begin{aligned} L - \frac{\partial U'}{\partial R} &= (L - \lambda)\left(1 + \frac{R}{AE}\right) \\ \frac{\partial U'}{\partial R} + \frac{R(L - \lambda)}{AE} &= \lambda \quad \dots \quad \text{Eq. 2.9} \end{aligned}$$

The strain energy of the member MN is $u = \frac{R^2(L - \lambda)}{2AE}$, so that

$$\frac{R(L - \lambda)}{AE} = \frac{\partial u}{\partial R}$$

Substituting this value in equation 2.9, we obtain

$$\frac{\partial U'}{\partial R} + \frac{\partial u}{\partial R} = \lambda$$

or
$$\frac{\partial}{\partial R}(U' + u) = \lambda$$

But $(U' + u)$ is the total strain energy of the frame, including the member MN, and may be written as U .

Hence,
$$\frac{\partial U}{\partial R} = \lambda \quad \dots \quad \text{Eq. 2.10}$$

i.e., the partial differential coefficient of the strain energy of a redundant structure with respect to the load in any member is equal to the initial lack of fit of that member.

The *principle of least work* or *minimum strain energy* is a special case of this theorem, obtained when $\lambda = 0$. If the member MN is initially of the correct length to fit between the joints M and N it follows from equation 2.10 that $\frac{\partial U}{\partial R} = 0$.

This means that the load R will be of such a value as to make the strain energy a minimum.

The loads in the members of a redundant frame may be determined by application of this principle as follows. Let there be n redundant members. Each of these n members is assumed removed from the frame and replaced by loads $R_1, R_2, \dots, R_n$ respectively at the appropriate joints, and the loads in the remaining members evaluated in terms of the R 's and the external loads. Next, the strain energy of the frame is calculated, and it is obvious that this will be a function of the R 's. By setting the partial differential coefficient of the strain energy with respect to each of the R 's in turn equal to zero or to the amount by which the particular member considered is too short initially, a sufficient number of simultaneous equations will be obtained to enable the R 's to be determined. The loads in all the other members then follow by substitution.

It may be noted here, although it should be obvious, that equation 2.10 applies only to a redundant structure, since it is only in such a structure that the insertion of a member, which is not of the correct initial length, will induce *self-straining*.

The examples which follow illustrate the application of the above to framed structures. It should be remarked, however, that the general principles apply to any redundant structure, and we shall see other applications in later chapters.

Example 2.5. In the frame of Fig. 2.11, the member BD is initially 0.1 in. too short. Determine the loads induced by forcing this member into place, given the following information. The members AB, BC, CD, DA are duralumin tubes, of cross-sectional area 0.25 sq. in., for which E is 10^7 lb. per sq. in. Members AC and BD are steel wires, of cross-sectional area 0.025 sq. in., for which the value of E is 30×10^6 lb. per sq. in.

We assume the member BD to be removed and replaced by loads R at B

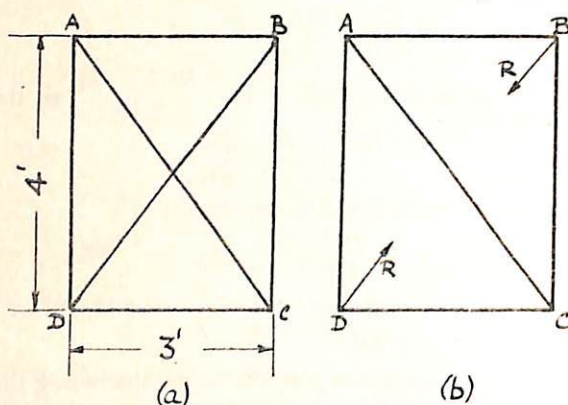


FIG. 2.11.—EXAMPLE 2.5.

and D, as indicated in Fig. 2.11 (b), the directions of these loads being such as to make R a tensile load in BD. The loads in all the members and the strain energy are evaluated in terms of R . Finally, if U is the total strain energy, including that of BD, we put $\frac{\partial U}{\partial R}$ equal to 0.1 in., and obtain an equation giving the value of R . It is again convenient to set out the calculations in a tabular form. A negative value for P indicates compression.

Member.	L in.	A sq. in.	P lb.	$\frac{P^2 L}{2AE}$	P lb.
AB	36	0.25	$-0.6R$	$25.92R^2 \times 10^{-7}$	-308
BC	48	0.25	$-0.8R$	$61.44R^2 \times 10^{-7}$	-410
CD	36	0.25	$-0.6R$	$25.92R^2 \times 10^{-7}$	-308
DA	48	0.25	$-0.8R$	$61.44R^2 \times 10^{-7}$	-410
AC	60	0.025	R	$400R^2 \times 10^{-7}$	513
BD	60	0.025	R	$400R^2 \times 10^{-7}$	513

By addition of the figures in the fifth column of this table, we have

$$U = 974.72R^2 \times 10^{-7}$$

Then,

$$\frac{\partial U}{\partial R} = 1949.44R \times 10^{-7} = 0.1$$

and

$$R = 513 \text{ lb.}$$

The loads in the members are given in the last column of the table.

2.11. Redundant Moments

As is the case with other strain energy theorems, the principle of least work can be applied to moments which can be regarded as redundancies. The strain energy in such cases will, of course, include that due to bending. Typical structures in which redundant internal moments arise are rigid jointed frames, and beams which are continuous over more than two supports. The general treatment of rigid jointed frames is beyond the scope of this book, and continuous beams will be discussed in a later chapter, where other means of treating the problem will be used.

2.12. Suddenly Applied Loads

In the preceding paragraphs, the loads and moments have been assumed to be applied gradually. If the application is sudden, the conditions existing instantaneously are different, and may be visualised somewhat as follows.

Suppose a tensile load P_1 is applied gradually to a bar and produces an extension δ_1 . Then the load-extension diagram is triangular as in Fig. 2.12 (a), the

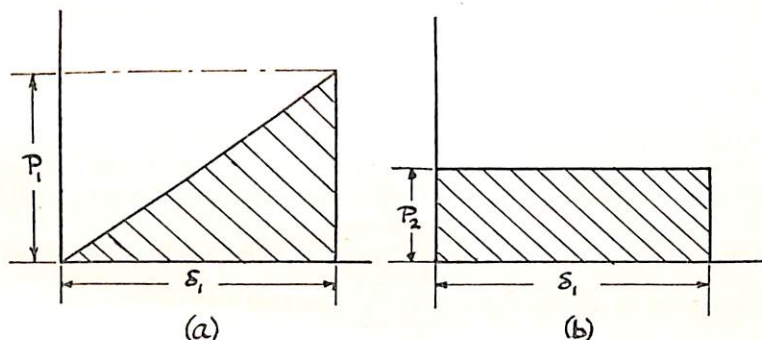


FIG. 2.12.—LOAD DISPLACEMENT DIAGRAMS FOR A TENSILE LOAD APPLIED
(a) GRADUALLY, (b) SUDDENLY.

elastic limit being not exceeded. If the sudden application of a load P_2 to an identical bar produces instantaneously the same extension δ_1 , the load-extension diagram will be rectangular, as in Fig. 2.12 (b). Since the bar extends the same amount in each case, the internal strain energy, and hence the external work done must be the same. It follows that the areas of the two load-extension diagrams are equal.

$$\text{i.e.,} \quad \frac{1}{2}P_1\delta_1 = P_2\delta_1$$

$$\text{or} \quad P_1 = 2P_2$$

This means that the gradually applied load must be twice as big as the suddenly applied load to produce the same extension.

Similarly, we may show that a suddenly applied load will produce instantaneously twice the extension given by the same load applied gradually, always presupposing that the elastic limit is not exceeded.

The bar which is loaded suddenly will oscillate about its equilibrium position, finally settling down with the extension which gradual application of the load would produce, but it must be realised that instantaneous strains, and hence stresses, of twice the static value will be reached.

Load dropped from a Height. If the suddenly applied load is that due to a
c

weight dropped from a height, the instantaneous deflection, and hence stress if required, may be determined from the fact that the strain energy absorbed must equal the total potential energy lost by the weight, since we may assume in general that the weight is instantaneously at rest at the position of maximum deflection, so that it has then no kinetic energy.

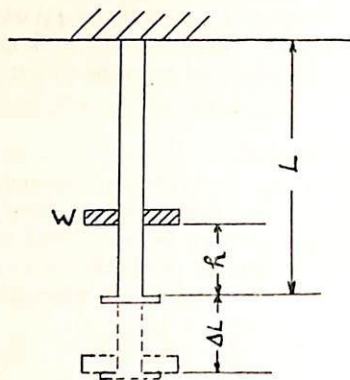


FIG. 2.13.—SUDDENLY APPLIED TENSILE LOAD (LOAD DROPPED FROM A HEIGHT).

For example, if a weight W can slide on a rod, of length L , rigidly held at its upper end, with a stop at its lower end, as in Fig. 2.13, then assuming the rod to extend a maximum amount ΔL when the weight is allowed to fall through a height h on to the stop, we may write,

$$\text{internal strain energy} = \text{potential energy lost by weight} = W(h + \Delta L) \quad \text{Eq. 2.11}$$

The instantaneous strain in the rod is $\frac{\Delta L}{L}$, so that the corresponding stress is $\frac{E\Delta L}{L}$, which would be produced by a static load of $\frac{AE\Delta L}{L}$, where A is the cross-sectional area of the rod. From equation 2.1, the strain energy is then given by

$$U = \left(\frac{AE\Delta L}{L} \right)^2 \cdot \frac{L}{2AE} = \frac{AE(\Delta L)^2}{2L} \quad \text{Eq. 2.12}$$

From equations 2.11 and 2.12,

$$\frac{AE(\Delta L)^2}{2L} = W(h + \Delta L) \quad \text{Eq. 2.13}$$

This is an equation from which the instantaneous strain may be determined. As an approximation, ΔL may generally be assumed to be small compared with h , so that

$$\Delta L \simeq \sqrt{\frac{2WLh}{AE}} \quad \text{Eq. 2.14}$$

Alternatively, if desired, equation 2.13 may be written in terms of the stress, p , instead of the extension ΔL , by making use of the relationship $p = \frac{E\Delta L}{L}$, giving

$$\frac{ALp^2}{2E} = W\left(h + \frac{Lp}{E}\right) \quad \text{Eq. 2.15}$$

Again, neglecting the extension in comparison with h , we have

$$p \simeq \sqrt{\frac{2WEh}{AL}} \quad \text{Eq. 2.16}$$

Sudden Loading on Beams. The problem of sudden loading of a beam may perhaps be best illustrated by the following example.

A weight W is dropped from a height h on to a uniform simply supported beam, of length $(a + b)$, at a distance a from one end. Determine the strain energy of the beam in terms of the deflection of the point of impact at any instant, and obtain an expression for the maximum deflection of the beam at the point of impact (A.F.R.Ae.S.).

We may obtain the strain energy of the beam in terms of δ , the deflection of the point of impact, by application of Castigliano's first theorem, as follows.

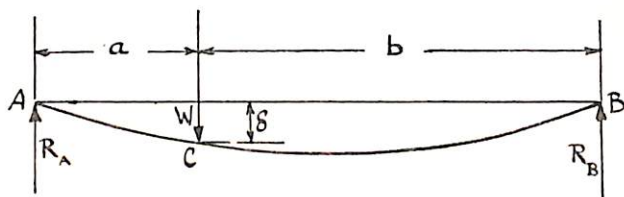


FIG. 2.14.—SUDDEN LOADING ON A BEAM.

Let the deflection δ correspond to a static (i.e. gradually applied) load P . Then, referring to Fig. 2.14,

$$R_A = \frac{Pb}{a+b}$$

and

$$R_B = \frac{Pa}{a+b}$$

$$\text{From A to C, the bending moment } M = R_A x = \frac{Pbx}{a+b}$$

$$\text{From B to C, the bending moment } M = R_B X = \frac{PaX}{a+b}$$

(x is distance measured from A, and X is distance measured from B. $X = a + b - x$)

The strain energy, U , of the bent beam is given by equation 2.3 as

$$\begin{aligned} U &= \int_0^L \frac{M^2 dx}{2EI} \\ &= \int_0^a \frac{(Pbx)^2 dx}{(a+b)^2 2EI} + \int_0^b \frac{(PaX)^2 dX}{(a+b)^2 2EI} \\ &= \frac{P^2}{2EI(a+b)^2} \left(b^2 \frac{a^3}{3} + a^2 \frac{b^3}{3} \right) \\ &= \frac{P^2 a^2 b^2}{6EI(a+b)} \quad \text{Eq. 2.17} \end{aligned}$$

Now by Castigliano's first theorem, $\frac{\partial U}{\partial P} = \delta$.

Hence,
$$\frac{Pa^2b^2}{3EI(a+b)} = \delta \quad \text{Eq. 2.18}$$

Eliminating P from equations 2.17 and 2.18, we obtain U in terms of δ . Thus,

$$U = \frac{3}{2} \cdot \frac{EI\delta^2(a+b)}{a^2b^2} \quad \text{Eq. 2.19}$$

The maximum deflection, Δ , will be reached when the internal strain energy is equal to the external work done, or the loss in potential energy of the weight.

From equation 2.19, when $\delta = \Delta$, the internal strain energy, U , is given by

$$U = \frac{3EI\Delta^2(a+b)}{2a^2b^2}$$

The external work done $= W(h + \Delta)$.

Hence, Δ is given by

$$\frac{3EI\Delta^2(a+b)}{2a^2b^2} = W(h + \Delta)$$

or

$$\frac{3EI(a+b)}{2Wa^2b^2}\Delta^2 - \Delta - h = 0$$

Solving this quadratic equation in the usual way, we obtain the maximum deflection at the point of impact as

$$\Delta = \frac{1 + \sqrt{\left[1 + \frac{6EIh(a+b)}{Wa^2b^2}\right]}}{\frac{3EI(a+b)}{Wa^2b^2}}$$

CHAPTER III

FUNDAMENTALS OF STRESS AND STRAIN

3.1. General

In Volume I, Chapter IX, the main types of stress and strain were defined, and in subsequent chapters the stresses due separately to axial load, bending and torsion were determined in terms of the applied loading and the sectional characteristics of the member considered. It will be appreciated that circumstances will often arise in which a member of a structure is subjected to "combined loading," such as a simultaneous application of bending and torsion. It is necessary to investigate the stress conditions in such circumstances, and an elementary approach to the problem will be covered in this chapter. The problem of combined stresses will be preceded by some fundamental ideas regarding stresses on planes inclined to the plane for which the applied stress is determined.

3.2. Complementary Shear Stress

Consider the equilibrium of a small element ABCD surrounding the point P in a block subjected to a shear force F as shown in Fig. 3.1. Let the shear stress

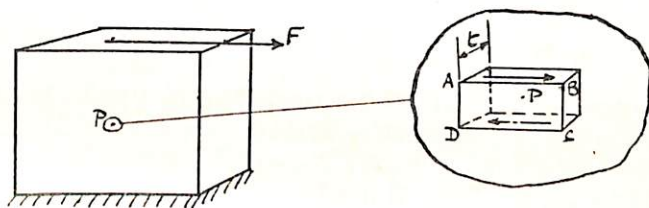


FIG. 3.1.—ELEMENT OF A BLOCK SUBJECTED TO SHEAR.

over the face AB be uniform and equal to q . Then the shear force on the face AB is $q \cdot AB \cdot t$. From considerations of equilibrium there will be an equal and opposite shear force on the face DC, and since the area of the face DC is equal to that of the face AB, the shear stress on DC will also be equal to q . These two shear forces constitute a couple, and to maintain equilibrium there must be another couple of equal magnitude and opposite sign, which will be provided by shear forces on the faces BC and DA. It is apparent that the shear stress, q_1 , will be the same on these two faces, since the two forces and the areas of the faces are equal. Since the two couples must balance we may write

$$(q_1 \cdot BC \cdot t) \cdot AB = (q \cdot AB \cdot t) \cdot BC$$

Hence

$$q_1 = q \quad \dots \dots \dots \text{Eq. 3.1}$$

It follows that a shear stress q in a given direction gives rise to an equal shear stress q in a direction at right-angles to that given. Such shear stresses are termed *complementary*.

3.3. Stresses on Oblique Planes

(a) Consider first the simple case of a member subjected to an axial tensile load, P . In Fig. 3.2 let the area of a normal cross-section, AB, be A . Then

the stress on the plane AB will be a tensile stress, normal to the plane, of magnitude $P/A = p$.

Now consider the section CD. If we imagine the member to be cut into

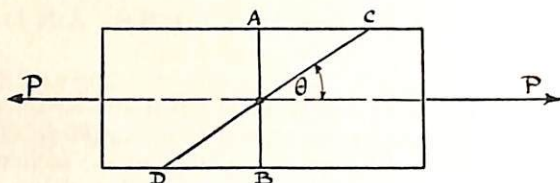


FIG. 3.2.

two pieces at CD, the action of the left-hand portion on the right-hand portion must be equivalent to the load P (see Fig. 3.3). P may be resolved into a com-

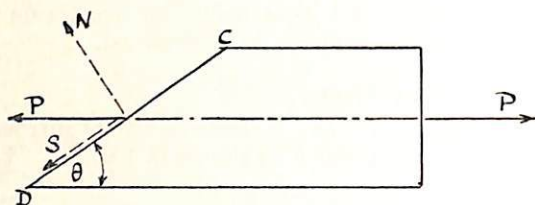


FIG. 3.3.—NORMAL AND SHEAR LOADS ON AN OBLIQUE PLANE.

ponent N , perpendicular to CD, and a component S , parallel to CD, such that

$$N = P \sin \theta$$

and

$$S = P \cos \theta$$

Then the normal stress on CD will be

$$f_n = N/\text{Area CD}$$

and the shear stress on CD will be

$$f_s = S/\text{Area CD}$$

Now

$$\text{Area CD}/\text{Area AB} = CD/AB = \text{cosec } \theta$$

so that

$$\text{Area CD} = A \text{ cosec } \theta$$

Hence

$$f_n = P \sin \theta / A \text{ cosec } \theta = p \sin^2 \theta \quad \text{Eq. 3.2}$$

and

$$f_s = P \cos \theta / A \text{ cosec } \theta = p \sin \theta \cos \theta = \frac{1}{2} p \sin 2\theta \quad \text{Eq. 3.3}$$

Consequently an axial tension gives rise to both normal and shear stresses on planes which are not perpendicular to the line of the load. It must be noted carefully that it is not correct to resolve stresses directly. The applied loads must be resolved into the required directions and the components so obtained divided by the appropriate areas.

From equation 3.3 it follows that f_s will have its maximum value when $\sin 2\theta = 1$, or when $\theta = 45^\circ$, and that this maximum is $p/2$. This explains why a specimen tested in tension often fails by shear along planes inclined at 45° to the line of load, as indicated in Fig. 3.4. In such cases the ultimate shear stress of the material is less than half of the ultimate tensile stress, so that the ultimate shear stress is reached on the planes of maximum shear before the ultimate tensile stress occurs on the normal cross-section.

(b) Considering now an element subjected to pure shear, as illustrated in Fig. 3.5, the magnitude of the shear stress being q , let us determine the normal and shear stresses on the plane BE. The element BCE must be in equilibrium



FIG. 3.4.—SHEAR FAILURE OF TENSILE TEST SPECIMEN.

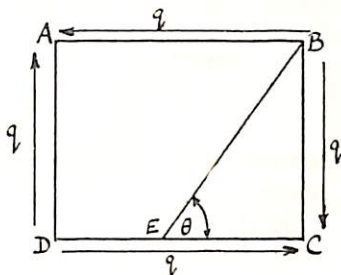


FIG. 3.5.—ELEMENT SUBJECTED TO PURE SHEAR.

under the stress system shown in Fig. 3.6. Resolving forces on this element normal to BE, and assuming unit thickness, we have

$$f_n \cdot BE - (q \cdot BC) \cos \theta - (q \cdot CE) \sin \theta = 0$$

Dividing through by BE, this becomes

$$f_n = q \sin \theta \cos \theta + q \cos \theta \sin \theta = q \sin 2\theta \quad \text{Eq. 3.4}$$

Resolving forces parallel to BE gives

$$f_s \cdot BE - (q \cdot BC) \sin \theta + (q \cdot CE) \cos \theta = 0$$

i.e.,

$$f_s = q \sin^2 \theta - q \cos^2 \theta = -q \cos 2\theta \quad \text{Eq. 3.5}$$

From equation 3.4, f_n is a maximum when $\sin 2\theta = 1$, or when θ is 45° , and the maximum value of f_n is q . Note that when $\theta = 135^\circ$, $\sin 2\theta = -1$, and f_n is then a compressive stress of magnitude q . For both these values of θ f_s is zero,

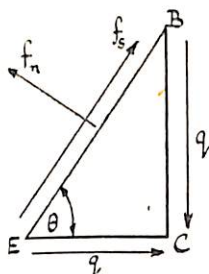


FIG. 3.6.—STRESS SYSTEM FOR TRIANGULAR ELEMENT OF BLOCK IN PURE SHEAR.

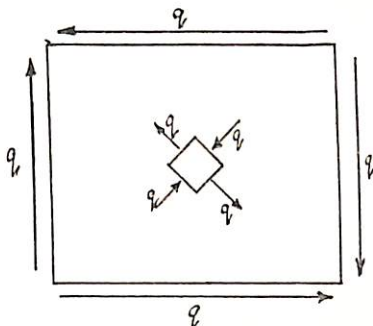


FIG. 3.7.—DIAGONAL TENSION AND COMPRESSION RESULTING FROM PURE SHEAR.

so that a state of pure shear produces tensile and compressive stresses, each of the same magnitude as the shear stress, on planes inclined respectively such that $\theta = 45^\circ$ and 135° . This result, which may be represented by Fig. 3.7, will be found to be of importance when we consider the behaviour of thin plates in shear in a later chapter.

Example 3.1. A bar, of cross-sectional area 0.5 sq. in., is subjected to an

axial tensile load of 4 tons. Determine the stresses on a plane inclined at an angle of 30° to the axis of the bar, and find the maximum shear stress which is developed.

Referring to Fig. 3.2, in this case the area of the plane AB is 0.5 sq. in., and $\theta = 30^\circ$, $P = 4$ tons.

Then area of plane CD = $0.5 \operatorname{cosec} 30^\circ = 1.0$ sq. in.

Component of force normal to CD = $4 \sin 30^\circ = 2$ tons

" " " parallel to CD = $4 \cos 30^\circ = 3.464$ tons

Hence, normal stress on CD = $2/1.0 = 2$ tons per sq. in.

shear " " CD = 3.464 tons per sq. in.

The maximum shear stress occurs on a plane inclined at 45° to the axis, and is given by

$$f_{s_{\max}} = \frac{4 \cos 45^\circ}{0.5 \operatorname{cosec} 45^\circ} = 4 \text{ tons per sq. in.}$$

3.4. Combined Stresses

With an element ABCD subjected to the direct stresses f_x , and f_y , and to the shear stress q as indicated in Fig. 3.8 (a), we investigate the stress system on the plane BE. This stress system may be represented by a normal stress f_n and a

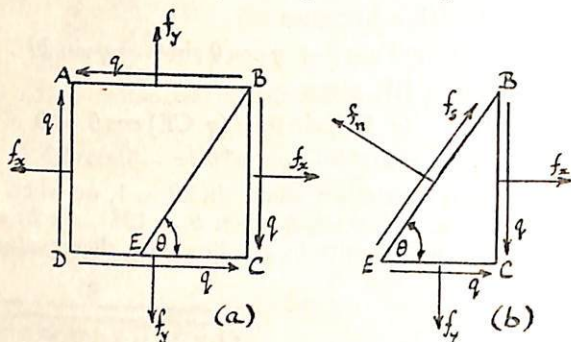


FIG. 3.8.—COMBINED STRESSES.

shear stress f_s (Fig. 3.8 (b)). From the conditions of equilibrium of the portion BCE, assuming unit thickness, we may write, resolving forces perpendicular to BE,

$$f_n \cdot BE - (f_x \cdot BC) \sin \theta - (f_y \cdot CE) \cos \theta - (q \cdot BC) \cos \theta - (q \cdot CE) \sin \theta = 0$$

Dividing through by BE this becomes

$$\begin{aligned} f_n &= f_x \sin^2 \theta + f_y \cos^2 \theta + 2q \sin \theta \cos \theta \\ &= f_x \left(\frac{1 - \cos 2\theta}{2} \right) + f_y \left(\frac{1 + \cos 2\theta}{2} \right) + q \sin 2\theta \\ &= \frac{f_x + f_y}{2} + \left(\frac{f_y - f_x}{2} \right) \cos 2\theta + q \sin 2\theta \end{aligned} \quad \text{Eq. 3.6}$$

Resolving parallel to BE,

$$f_s \cdot BE + (f_x \cdot BC) \cos \theta - (f_y \cdot CE) \sin \theta - (q \cdot BC) \sin \theta + (q \cdot CE) \cos \theta = 0$$

$$\begin{aligned} f_s &= -f_x \sin \theta \cos \theta + f_y \sin \theta \cos \theta - q(\cos^2 \theta - \sin^2 \theta) \\ &= \left(\frac{f_y - f_x}{2} \right) \sin 2\theta - q \cos 2\theta \end{aligned} \quad \text{Eq. 3.7}$$

Equations 3.6 and 3.7 give the normal and shear stresses respectively on a plane inclined at θ to the direction of f_x . By assuming the element to shrink to infinitesimal dimensions we have the stress system at a point. It is apparent that values of θ may exist for which f_n is greater than either f_x or f_y and f_s greater than q . For purposes of design we obviously require to know the maximum values of f_n and f_s , and in the next sections we proceed to the evaluation of these maxima.

3.5. Principal Stresses and Principal Planes

Equation 3.6 gives f_n as a function of θ , and we know that when f_n is a maximum or minimum then $\frac{df_n}{d\theta} = 0$. Hence, differentiating equation 3.6 with respect to θ , we obtain the condition for maximum or minimum f_n as

$$-(f_y - f_x) \sin 2\theta + 2q \cos 2\theta = 0$$

$$\text{i.e.,} \quad \tan 2\theta = \frac{2q}{f_y - f_x} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{Eq. 3.8}$$

This gives two values of 2θ which differ by 180° , or two values of θ , say θ_1 and θ_2 , which differ by 90° , so that $\theta_2 = \theta_1 + 90^\circ$. On one of the planes so determined f_n is a maximum, on the other it is a minimum. Let the values of f_n corresponding to θ_1 and θ_2 be f_1 and f_2 respectively. From equation 3.8, we may write

$$\sin 2\theta_1 = \frac{2q}{\sqrt{\{(f_y - f_x)^2 + 4q^2\}}}$$

$$\cos 2\theta_1 = \frac{f_y - f_x}{\sqrt{\{(f_y - f_x)^2 + 4q^2\}}}$$

$$\sin 2\theta_2 = \frac{-2q}{\sqrt{\{(f_y - f_x)^2 + 4q^2\}}}$$

$$\cos 2\theta_2 = \frac{-(f_y - f_x)}{\sqrt{\{(f_y - f_x)^2 + 4q^2\}}}$$

These may be combined into

$$\left. \begin{aligned} \sin 2\theta_{1,2} &= \pm \frac{2q}{\sqrt{\{(f_y - f_x)^2 + 4q^2\}}} \\ \cos 2\theta_{1,2} &= \pm \frac{(f_y - f_x)}{\sqrt{\{(f_y - f_x)^2 + 4q^2\}}} \end{aligned} \right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{Eqs. 3.9}$$

Inserting these values in equation 3.6 we obtain

$$\begin{aligned} f_{1,2} &= \frac{f_x + f_y}{2} \pm \frac{f_y - f_x}{2} \frac{f_y - f_x}{\sqrt{\{(f_y - f_x)^2 + 4q^2\}}} \pm q \frac{2q}{\sqrt{\{(f_y - f_x)^2 + 4q^2\}}} \\ &= \frac{f_x + f_y}{2} \pm \frac{1}{2} \frac{(f_y - f_x)^2 + 4q^2}{\sqrt{\{(f_y - f_x)^2 + 4q^2\}}} \\ &= \frac{f_x + f_y}{2} \pm \frac{1}{2} \sqrt{(f_y - f_x)^2 + 4q^2} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{Eq. 3.10 (a)} \end{aligned}$$

$$= \frac{f_x + f_y}{2} \pm \sqrt{\left(\frac{f_y - f_x}{2}\right)^2 + q^2} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \text{Eq. 3.10 (b)}$$

Equations 3.10 give alternative forms for the maximum and minimum values of f_n . The + sign corresponds to $f_{n_{\max}}$, the - sign to $f_{n_{\min}}$. From equations 3.7 and 3.9 it will be seen easily that when f_n is a maximum or minimum f_s is zero. Hence there exist two planes at right angles to each other on which the normal stress is a maximum or minimum and the shear stress is zero. These planes are called the *principal planes*, and the normal stresses on them are the *principal stresses*.

If f_y is zero, equations 3.10 reduce to the simple form

$$f_{1,2} = \frac{f_x}{2} \pm \sqrt{\left\{\frac{f_x}{2}\right\}^2 + q^2} \quad \text{Eq. 3.11}$$

3.6. Maximum Shear Stress

The condition for maximum shear stress is $\frac{df_s}{d\theta} = 0$, and from equation 3.7 this is

$$(f_y - f_x) \cos 2\theta + 2q \sin 2\theta = 0$$

i.e.,

$$\tan 2\theta = -\frac{f_y - f_x}{2q} \quad \text{Eq. 3.12}$$

This equation again gives two values of θ , say θ_3 and θ_4 , which differ by 90° . From equations 3.9 and 3.12 we note that

$$\tan 2\theta_1 \cdot \tan 2\theta_3 = -1$$

Hence $2\theta_1$ and $2\theta_3$ differ by 90° , so that θ_1 and θ_3 differ by 45° . Consequently, maximum shear stress occurs on planes inclined at 45° to the principal planes.

The value of the maximum shear stress is obtained, by substituting for $\sin 2\theta$ and $\cos 2\theta$ from equation 3.12 into equation 3.7, as

$$f_{s_{\max}} = \pm \frac{1}{2} \sqrt{(f_y - f_x)^2 + 4q^2} \quad \text{Eq. 3.13 (a)}$$

or

$$f_{s_{\max}} = \pm \sqrt{\left\{\frac{f_y - f_x}{2}\right\}^2 + q^2} \quad \text{Eq. 3.13 (b)}$$

If f_y is zero, this reduces to

$$f_{s_{\max}} = \pm \sqrt{\left\{\frac{f_x}{2}\right\}^2 + q^2} \quad \text{Eq. 3.14}$$

A numerical example will illustrate the necessity for calculating the principal stresses and the maximum shear stress in cases of combined loading.

Example 3.2. A bolt, of cross-sectional area 0.5 sq. in., is subjected to a tensile load of 5000 lb. and a shear force of 2500 lb. Determine the principal stresses and the maximum shear stress.

In this case, $f_x = \frac{5000}{0.5} = 10,000$ lb. per sq. in.

$$q = \frac{2500}{0.5} = 5000 \quad \text{,, , , ,}$$

Hence,

$$f_1 = \frac{f_x}{2} + \sqrt{\left(\frac{f_x}{2}\right)^2 + q^2} = 5000 + \sqrt{(5000)^2 + (5000)^2} = 12,070 \text{ lb. per sq. in.}$$

$$f_2 = \frac{f_x}{2} - \sqrt{\left(\frac{f_x}{2}\right)^2 + q^2} = 5000 - \sqrt{(5000)^2 + (5000)^2} = -2070 \text{ lb. per sq. in.}$$

$$f_{s_{\max.}} = \sqrt{\left(\frac{f_x}{2}\right)^2 + q^2} = \sqrt{(5000)^2 + (5000)^2} = 7070 \text{ lb. per sq. in.}$$

The negative value obtained for f_2 indicates that this is a compressive stress. If this bolt were designed to the apparent stresses f_x and q , it would obviously not be strong enough, since tensile and shear stresses considerably greater than f_x and q are developed.

3.7. The Circle of Stress

A convenient graphical method of determining the principal stresses and the maximum shear stress at a point for a given stress system is that known as Mohr's Circle or Culman's Circle. Referring to equations 3.6 and 3.7, we may eliminate θ between these equations as follows. We rewrite the equations as under,

$$f_n - \frac{f_x + f_y}{2} = \frac{(f_y - f_x)}{2} \cos 2\theta + q \sin 2\theta$$

and

$$f_s = \frac{(f_y - f_x)}{2} \sin 2\theta - q \cos 2\theta$$

Squaring each of these two equations and then adding, we have

$$\left\{ f_n - \frac{f_x + f_y}{2} \right\}^2 + f_s^2 = \left(\frac{f_y - f_x}{2} \right)^2 + q^2. \quad \text{Eq. 3.15}$$

For a given stress system, f_x , f_y , and q , the normal and shear stresses f_n and f_s on any plane must satisfy equation 3.15. If this equation is plotted with f_n and f_s as orthogonal co-ordinates, the resulting curve is a circle, with its centre at the point $\left(\frac{f_x + f_y}{2}, 0 \right)$ on the axis of f_n , and of radius equal to $\sqrt{\left(\frac{f_y - f_x}{2} \right)^2 + q^2}$.

The principal stresses are given by the maximum values of f_n , and the maximum shear stress is given by the maximum value of f_s . It is possible to determine

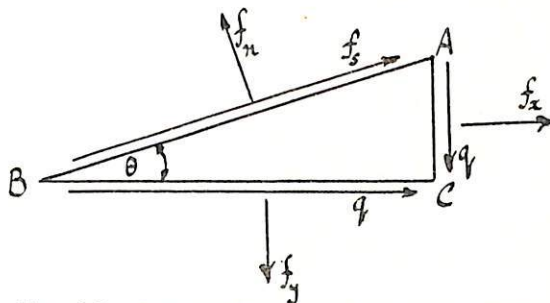


FIG. 3.9.—STRESS SYSTEM ON ELEMENTARY PRISM.

from this circle the normal and shear stresses on planes inclined in any required directions, and this general use of the method will be described. Let the stress system on an elementary prism, of unit thickness, be as shown in Fig. 3.9. f_x ,

f_y and q are assumed known, whilst f_n and f_s are to be determined. The forces on the three faces AB, AC, BC may be expressed as follows :—

Normal force on AC	$= f_x \cdot AC$
Shear " " AC	$= q \cdot AC$
Normal " " BC	$= f_y \cdot BC = f_y \cdot AC \cot \theta$
Shear " " BC	$= q \cdot BC = q \cdot AC \cot \theta$
Normal " " AB	$= f_n \cdot AB = f_n \cdot AC \operatorname{cosec} \theta$
Shear " " AB	$= f_s \cdot AB = f_s \cdot AC \operatorname{cosec} \theta$

The construction of the circle is as follows (Fig. 3.10) :—Take two perpendicular axes, OX and OY. On OX mark off OD = f_x , and OE = f_y to a suitable

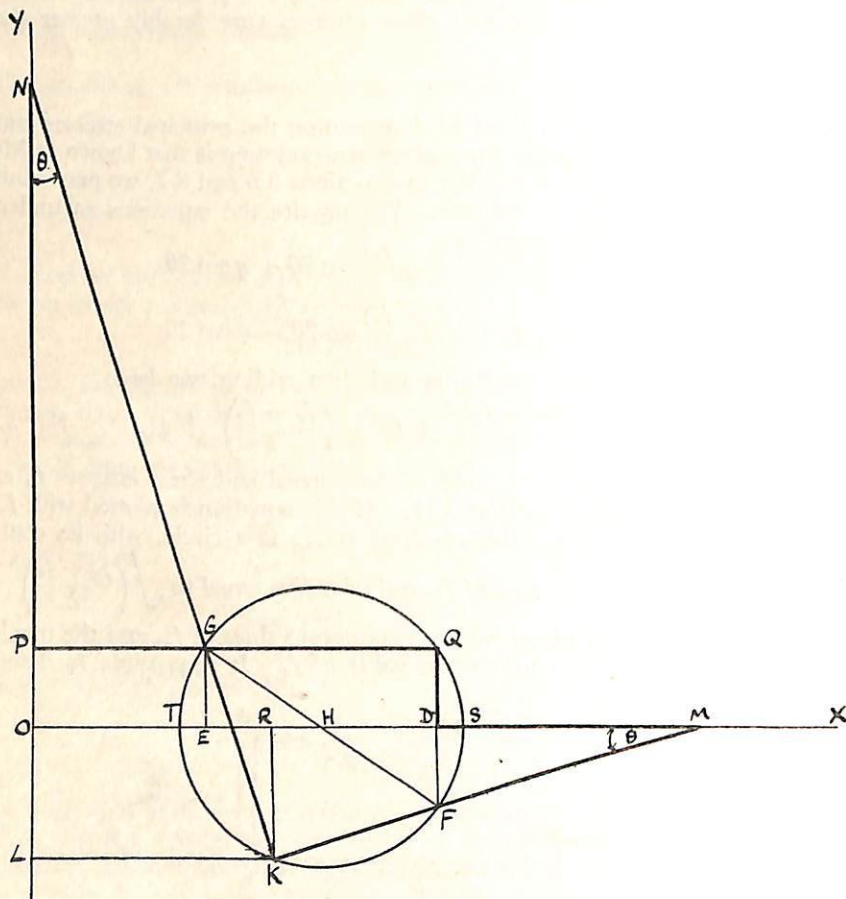


FIG. 3.10.—CONSTRUCTION AND PROOF FOR CIRCLE OF STRESS.

scale, measuring to the right for tension, and to the left for compression. At D draw DF = q at right-angles to OD and in the direction of the shear force on the face AC (downwards in our case). At E draw EG = DF, also perpendicular to OD but in the opposite direction from DF. Draw a circle on GF as diameter. Draw FK parallel to AB, meeting the circle again in K. Then

the point K gives the stress system on the plane AB, so that if KL is drawn perpendicular to OY, the length $KL = f_n$, and $OL = f_s$. That this is so may be demonstrated simply as follows. Let KF, produced if necessary, meet OD in M. Then the angle $FMD = \theta$. Let KG meet OY in N. Since the angle GKF is the angle in a semi-circle, NK is perpendicular to FK. Further, OY is perpendicular to OM. Hence the angle $KNL = \theta$. FD is produced to meet the circle again in Q, and obviously $QD = DF = q$. The line QG meets OY in P, and will be parallel to DO, so that $QP = DO = f_x$, and $PG = OE = f_y$. We can now note the following relationships,

$$NP = PG \cot \theta = f_y \cot \theta = \frac{\text{Normal force on BC}}{AC}$$

$$PQ = f_x = \frac{\text{Normal force on AC}}{AC}$$

$$QD = q = \frac{\text{Shear force on AC}}{AC}$$

$$DM = DF \cot \theta = q \cot \theta = \frac{\text{Shear force on BC}}{AC}$$

Consequently the lines NP, PQ, QD, DM represent the known forces in magnitude (the scale being the same for all) and in direction. Now the line MK is parallel to the shear force on AB and the line KN is parallel to the normal force on AB. Hence the polygon NPQDMKN must be the polygon of forces for the element ABC, and we have

$$MK = \frac{\text{Shear force on AB}}{AC}$$

$$NK = \frac{\text{Normal force on AB}}{AC}$$

Therefore,

$$MK = f_s \operatorname{cosec} \theta$$

and

$$NK = f_n \operatorname{cosec} \theta$$

Drawing KR perpendicular to OX, we note that the angle $RKG = \theta$, so that $MK = KR \operatorname{cosec} \theta = OL \operatorname{cosec} \theta$, and $NK = KL \operatorname{cosec} \theta$

$$\text{i.e.,} \quad OL = f_s$$

$$\text{and} \quad KL = f_n$$

The maximum value of f_n will obviously be OS, and for this value of f_n we observe that f_s is zero. The minimum value of f_n is OT, and again f_s is zero. Hence OS and OT give the principal stresses. It may be verified quite easily from the geometry of the figure that the lengths of OS and OT expressed in terms of f_x , f_y and q agree with the expressions previously derived for the principal stresses.

The maximum shear stress is the maximum ordinate of a point on the circle and hence is numerically equal to the radius of the circle.

The inclinations of the principal planes to the direction of f_x are given by the angles FSO and FTO.

In Fig. 3.11 if VHW is the diameter of the circle parallel to OY, HV and HW are both equal to the maximum shear stress, and the planes on which this shear stress occurs are parallel to FV and FW. It is seen that the angles VFS

and TFW are both 45° , so that the planes of maximum shear stress are at 45° to the principal planes, as proved earlier.

Three numerical examples illustrating the use of this method follow. In one

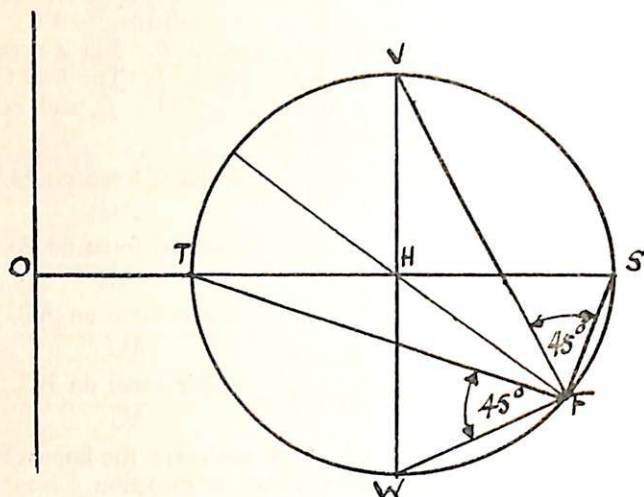


FIG. 3.11.—INCLINATION OF PLANES OF MAXIMUM SHEAR TO PRINCIPAL PLANES.

of these examples f_x and f_y are both tensile stresses, in another f_x is tensile and f_y is compressive, and in the third f_y is zero. In each example we are required

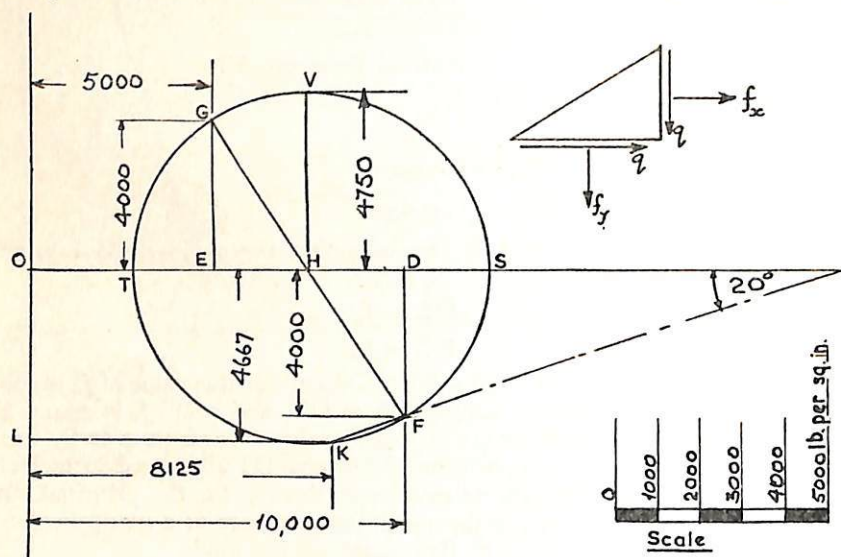


FIG. 3.12.

to determine the principal stresses and the maximum shear stress, and to find also the normal and shear stresses on a plane inclined at 20° to the direction of f_x .

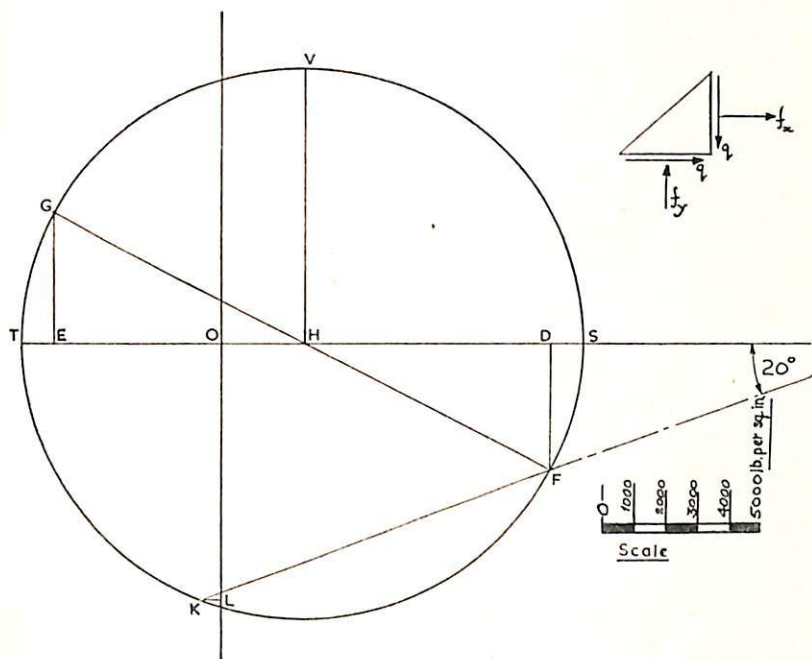


FIG. 3.13.

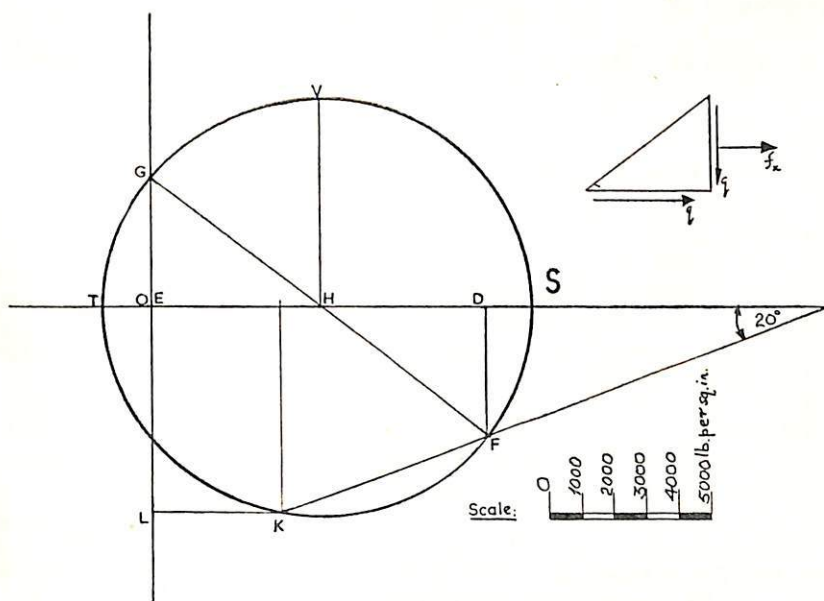


FIG. 3.14.

Example 3.3. $f_x = 10,000$ lb. per sq. in. tensile
 $f_y = 5000$ lb. per sq. in. tensile
 $q = 4000$ lb. per sq. in.

The stress circle, drawn as described above, is shown in Fig. 3.12. The required quantities are obtained by scaling off the corresponding lengths, and we determine,

Maximum principal stress = OS = 12,170 lb. per sq. in. tensile

Minimum " " = OT = 2835 lb. per sq. in. tensile

Maximum shear " = HV = 4750 lb. per sq. in.

On the plane at 20° to f_x , $f_n = \text{KL} = 8125$ lb. per sq. in. tensile
 $f_s = \text{OL} = 4667$ lb. per sq. in.

Example 3.4. $f_x = 10,000$ lb. per sq. in. tensile
 $f_y = 5000$ lb. per sq. in. compressive
 $q = 4000$ lb. per sq. in.

From the diagram (Fig. 3.13),

Maximum principal stress = OS = 10,920 lb. per sq. in. tensile

Minimum " " = OT = 5920 lb. per sq. in. compressive

Maximum shear " = HV = 9375 lb. per sq. in.

On the plane at 20° to f_x , $f_n = \text{KL} = 585$ lb. per sq. in. compressive
 $f_s = \text{OL} = 7835$ lb. per sq. in.

Example 3.5. $f_x = 10,000$ lb. per sq. in. tensile
 $q = 4000$ lb. per sq. in.

From the diagram (Fig. 3.14),

Maximum principal stress = OS = 11,500 lb. per sq. in. tensile

Minimum " " = OT = 1375 lb. per sq. in. compressive

Maximum shear " = HV = 6445 lb. per sq. in.

On the plane at 20° to f_x , $f_n = \text{KL} = 3917$ lb. per sq. in. tensile
 $f_s = \text{OL} = 6417$ lb. per sq. in.

In the above examples it will be noted that in Fig. 3.13 OE, representing f_y , is drawn to the left as this stress is compressive, and in Fig. 3.14, OE is zero as f_y is zero in this case.

3.8. Allowable Stresses under Combined Loading

Many theories have been put forward to determine the stress conditions at failure under combined loading, but there is no generally applicable theory. In most cases it is desirable to resort to structural tests for information regarding the allowable stresses for a particular component. This applies particularly for thin metal structures, where failure may be due to instability rather than material failure. In cases where buckling or instability does not occur, the allowable stress may be determined according to several different theories: a particular value of the maximum principal stress may be taken as the criterion, or a particular value of the maximum principal strain; a theory which has received wide acceptance for ductile materials is that which assigns an allowable value to the maximum shear stress. Generally these theories assume failure to occur at the yield point, not necessarily at fracture. A method which has been proposed by F. R. Shanley appears to have considerable promise, particularly in cases where failure may be due to instability. It is suggested in this method that the conditions for failure can be expressed by the equation

$$R_c^2 + R_b^2 + R_s^2 = 1$$

where R = stress ratio = applied stress/allowable stress, and the exponents α, β, γ are determined experimentally. The suffices c, b, s refer to compression, bending, and shear, and other stress conditions could be introduced similarly. The allowable stress used in calculating the stress ratio is the allowable stress for that particular type of stress acting alone. This method obviously gives correct results for one loading condition alone, and the implication is that it will give reasonably good results when one loading condition predominates. The actual values of α, β, γ to be used will depend on the loading conditions, varying with different combinations of loading. In cases where no experimental data exist to give values of the exponents, it is suggested that it will usually be safe to use the value one for all the exponents. For circular section tubes subjected to combined bending and torsion experiment indicates that the relationship to be used should be

$$R_b^2 + R_s^2 = 1$$

3.9. Relationships between the Elastic Constants of a Material

The quantities referred to as the elastic constants of a material are Young's Modulus (E), the Modulus of Rigidity (G), Poisson's Ratio (σ), and the Bulk Modulus (K). The first three of these have already been defined (I.9.6 to I.9.9) and the relation between them has been quoted, but not derived (I.9.10). The bulk modulus, K , is defined as follows: if a body, whose volume is V , is loaded by three equal stresses in three mutually perpendicular directions, and suffers a change in volume of amount δV , the bulk modulus is defined as the ratio of

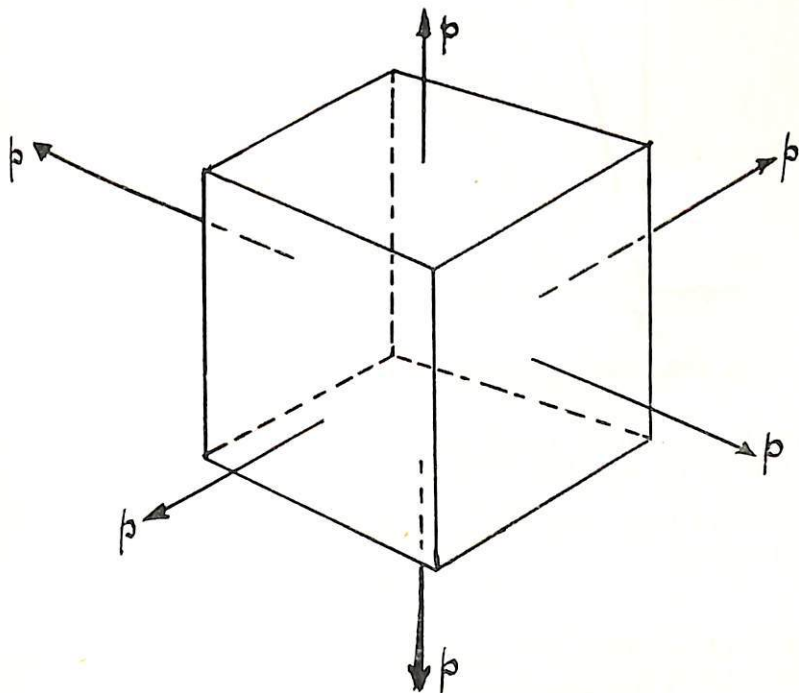


FIG. 3.15.—DEFINITION OF BULK MODULUS.

the stress to the volumetric strain, the latter quantity being $\delta V/V$. Thus for a stress, or pressure, of magnitude p , the bulk modulus is

$$K = \frac{p}{\frac{\delta V}{V}} \quad \text{Eq. 3.16}$$

If we consider a cube of side L loaded by three equal tensile stresses, p , as indicated in Fig. 3.15, so that each edge increases in length by δL , we have

$$\delta V = (L + \delta L)^3 - L^3 = 3L^2 \delta L + 3L(\delta L)^2 + (\delta L)^3$$

If δL is small we may neglect terms containing squares and higher powers of δL , so that

$$\delta V \simeq 3L^2 \delta L \quad \text{Eq. 3.17}$$

(This equation may also be obtained quite simply by differentiating the equation $V = L^3$ to give $dV = 3L^2 dL$.) The volumetric strain follows as

$$\frac{\delta V}{V} = \frac{3L^2 \delta L}{L^3} = \frac{3 \delta L}{L} = 3 \times \text{linear strain of one edge}$$

Then

$$K = p / \frac{\delta V}{V} = \frac{pL}{3\delta L} \quad \text{Eq. 3.18}$$

We can now proceed to determine the relations between the elastic constants.

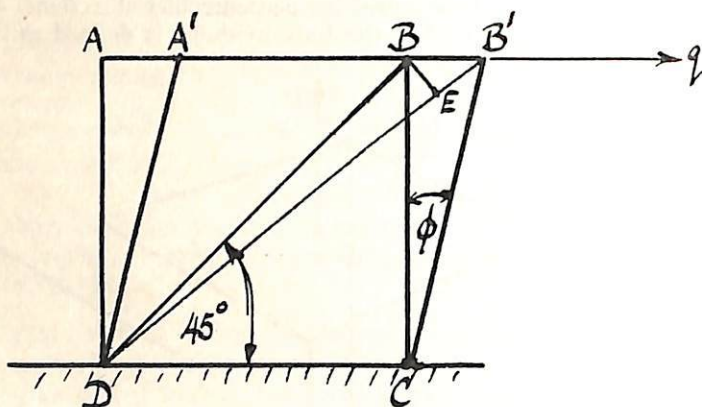


FIG. 3.16.—TENSILE STRAIN OF DIAGONAL FOR ELEMENT IN SHEAR.

If an element ABCD, initially square, is subjected to simple shear, the distortion may be represented by A'B'CD, and if BE is drawn perpendicular to B'D, then for small values of the shear strain, ϕ , the extension of the diagonal BD will be given by B'E (Fig. 3.16). Hence the linear strain (tensile) along the diagonal BD is given by

$$\epsilon = \frac{B'E}{BD}$$

But $B'E = BB' \cos BB'E$, and for small values of ϕ , the angle $BB'E \simeq 45^\circ$, so that $B'E \simeq BB' / \sqrt{2}$.

Furthermore, $BB' = BC \tan \phi \simeq BC \cdot \phi = BC \cdot q/G$

Therefore, $B'E \simeq \frac{BC \cdot q}{G\sqrt{2}}$

and $\epsilon = \frac{BC \cdot q}{BD \cdot G\sqrt{2}} = \frac{q}{2G} \left(\text{since } \frac{BC}{BD} = \sin 45^\circ = \frac{1}{\sqrt{2}} \right)$ Eq. 3.19

Now we know from para. 3.3 (b) that pure shear produces tensile and compressive stresses, equal in magnitude to the shear stress, on planes at 45° to the planes on which the shear stress is applied. Hence there is a tensile stress q along BD and a compressive stress q along AC.

Due to the tensile stress q along BD, the tensile strain of BD = q/E
 „ „ „ compressive „ q „ AC, „ „ „ „ BD = $\sigma q/E$

It follows that the total tensile strain, ϵ , of BD is given by

$$\epsilon = \frac{q}{E} + \frac{\sigma q}{E} = \frac{q}{E}(1 + \sigma) \quad \text{Eq. 3.20}$$

Equating the values for ϵ given by equations 3.19 and 3.20,

$$\frac{q}{2G} = \frac{q}{E}(1 + \sigma)$$

i.e., $G = \frac{E}{2(1 + \sigma)} \quad \text{Eq. 3.21}$

Now consider a cube loaded by equal tensile stresses on each of its faces, as in Fig. 3.15. The tensile strain, ϵ , of an edge will be

$$\epsilon = \frac{p}{E} - \frac{\sigma p}{E} - \frac{\sigma p}{E} = \frac{p}{E}(1 - 2\sigma)$$

But we have shown that the volumetric strain in such a case is three times the linear strain of one edge, so that we may write

$$\frac{\delta V}{V} = \frac{3p}{E}(1 - 2\sigma)$$

Using equation 3.16, this becomes,

$$\frac{p}{K} = \frac{3p}{E}(1 - 2\sigma)$$

or $K = \frac{E}{3(1 - 2\sigma)} \quad \text{Eq. 3.22}$

Equations 3.21 and 3.22 may be combined to give the following alternative relationships.

$$E = \frac{9KG}{3K + G}$$

$$K = \frac{GE}{3(3G - E)}$$

$$G = \frac{3EK}{9K - E}$$

$$\sigma = \frac{3K - 2G}{6K + 2G}$$

It should be noted that it has been assumed that the material is isotropic (i.e., that it has the same properties in all directions).

CHAPTER IV

EXTENSIONS TO THE THEORY OF SIMPLE BENDING, UNSYMMETRICAL BENDING, AND CURVED BEAMS

4.1. Unsymmetrical Bending

IN our original discussion of the theory of simple bending, (Volume I, Chapter XI), we placed no restriction on the axes about which bending was assumed to take place, other than saying that such an axis (the neutral axis) passed through the centroid of the cross-section. We shall now consider whether any further restriction is necessary.

Consider the section shown in Fig. 4.1, which is subjected to a bending moment M_x acting about *any* axis XX passing through the centroid of the section, G. Let YGY be an axis perpendicular to XX, both axes lying in the plane of

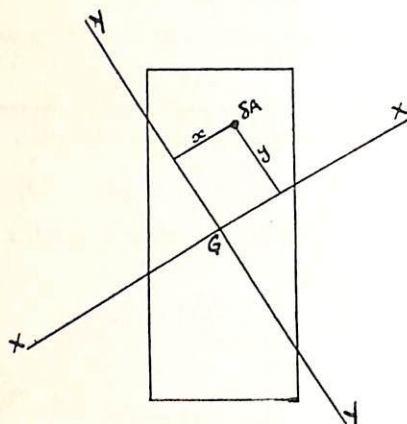


FIG. 4.1.

the section. Applying the simple bending formula developed in our previous consideration of the problem would yield a direct stress, f , on an element of area δA , located by co-ordinates x and y as shown, given by

$$f = \frac{M_x \cdot y}{I_x} \quad \dots \dots \dots \text{Eq. 4.1}$$

The end load on this element would then be

$$P = f \cdot \delta A = \frac{M_x}{I_x} \cdot y \delta A \quad \dots \dots \dots \text{Eq. 4.2}$$

The internal forces on the section must be in equilibrium with the external forces, the latter forming only a couple, M_x , about the XX axis. When considering the theory of simple bending previously, we used only two conditions of equilibrium, namely

$$\Sigma P = 0, \quad \text{and} \quad \Sigma M_x = 0$$

In fact, there are six conditions of equilibrium to be satisfied, since we are dealing here with forces in three dimensions. Since the only forces arising are perpendicular to the plane of the section, resolution of forces in directions parallel to the XX and YY axes will not give any equations of any meaning. (In each case we have simply $0 = 0$.) Similarly, equating to zero the sum of the moments about an axis ZZ, perpendicular to XX and YY, also gives $0 = 0$. The remaining condition to be satisfied is that the sum of the moments about YY must be zero. Referring to the figure, we see that the moment about YY of the end load on the element δA is given by

$$M_y = P \cdot x = \frac{M_x y \delta A x}{I_x} = \frac{M_x \cdot xy \delta A}{I_x}$$

Hence,

$$\Sigma M_y = \frac{M_x}{I_x} \Sigma xy \delta A \quad . \quad . \quad . \quad \text{Eq. 4.3}$$

and this will be zero only if

$$\Sigma xy \delta A = 0 \quad . \quad . \quad . \quad \text{Eq. 4.4}$$

It follows that we can apply the theory of simple bending in the form previously considered to determine the stresses due to bending only if the axis about which bending is assumed to be applied is such that equation 4.4 is satisfied. The quantity $\Sigma xy \delta A$ is termed the *product of inertia* about the axes XX and YY, and if this is zero for a particular pair of perpendicular axes through the centroid, these axes are called the *principal axes* of the section.

If bending takes place about any axis or axes other than the principal axes it is not possible to apply the simple bending formula directly and then add the stresses algebraically. The bending moments must be resolved into moments about the principal axes, and the stresses determined by application of the simple bending formula for bending about these axes separately may be added algebraically to give the resultant stresses. It is desirable, therefore, to indicate how the principal axes and the second moments of area about them may be determined.

4.2. Principal Axes

Let OX and OY be two perpendicular axes which are not principal axes, and let I_x , I_y and I_{xy} be the second moments of area and the product of inertia about these two axes. Let OX' and OY' be two other perpendicular axes such that the angle X'OX is θ (Fig. 4.2). Let an element δA be located at P, its

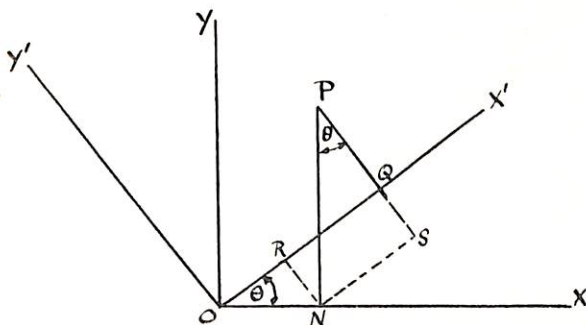


FIG. 4.2.—DETERMINATION OF PRINCIPAL AXES.

co-ordinates being (x, y) referred to the axes OX, OY and (x', y') referred to OX', OY' . Then in Fig. 4.2, $PN = y, PQ = y', ON = x, OQ = x'$.

$$x' = OQ = OR + RQ = OR + NS = x \cos \theta + y \sin \theta \quad \text{Eq. 4.5}$$

$$y' = PQ = PS - QS = PS - NR = y \cos \theta - x \sin \theta \quad \text{Eq. 4.6}$$

The product of inertia, $I_{x'y'}$, about OX' and OY' is given by

$$I_{x'y'} = \Sigma x'y'\delta A$$

Substituting the values of x' and y' given by equations 4.5 and 4.6,

$$\begin{aligned} I_{x'y'} &= \Sigma (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta)\delta A \\ &= \Sigma xy\delta A(\cos^2 \theta - \sin^2 \theta) - \Sigma x^2\delta A \cos \theta \sin \theta + \Sigma y^2\delta A \sin \theta \cos \theta \\ &= \cos 2\theta \Sigma xy\delta A - \frac{\sin 2\theta}{2} \Sigma x^2\delta A + \frac{\sin 2\theta}{2} \Sigma y^2\delta A \\ &= \cos 2\theta \cdot I_{xy} - \frac{\sin 2\theta}{2}(I_y - I_x) \quad \text{Eq. 4.7} \end{aligned}$$

Similarly, we may obtain the following expressions for the second moments of area about OX' and OY' .

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \quad \text{Eq. 4.8}$$

$$I_{y'} = \frac{I_x + I_y}{2} + \frac{I_y - I_x}{2} \cos 2\theta + I_{xy} \sin 2\theta \quad \text{Eq. 4.9}$$

If OX' and OY' are principal axes, then by definition $I_{x'y'}$ is zero, and from equation 4.7 it follows that

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} \quad \text{Eq. 4.10}$$

Equation 4.10 gives two values of 2θ differing by 180° , and hence two values of θ differing by 90° —i.e., it gives the angles $X'OX$ and $Y'OX$ for which OX' and OY' are principal axes.

From equation 4.10 we obtain

$$\begin{aligned} \sin 2\theta &= \frac{2I_{xy}}{\sqrt{(I_y - I_x)^2 + 4I_{xy}^2}} \\ \cos 2\theta &= \frac{I_y - I_x}{\sqrt{(I_y - I_x)^2 + 4I_{xy}^2}} \end{aligned} \quad \text{Eqs. 4.11}$$

and

Substituting these values in equations 4.8 and 4.9, we derive the second moments of area about the principal axes (principal moments of inertia) as

$$\begin{aligned} I_{x'} &= \frac{I_x + I_y}{2} - \frac{\sqrt{(I_y - I_x)^2 + 4I_{xy}^2}}{2} \\ &= \frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_y - I_x}{2}\right)^2 + I_{xy}^2} \quad \text{Eq. 4.12} \end{aligned}$$

$$I_{y'} = \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_y - I_x}{2}\right)^2 + I_{xy}^2} \quad \text{Eq. 4.13}$$

The similarity between these last equations and the expressions for principal stresses derived in Chapter III will be apparent.

Example 4.1. Locate the principal axes of the section shown in Fig. 4.3 and calculate the principal moments of inertia.

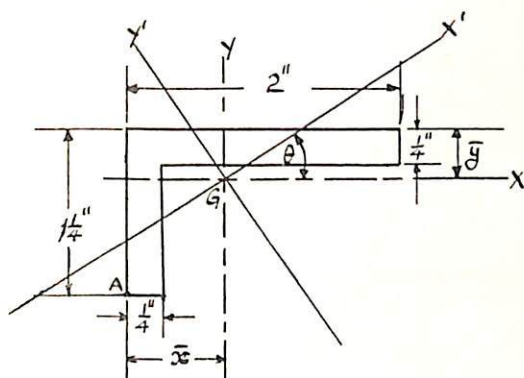


FIG. 4.3.—EXAMPLE 4.1.

Let G be the centroid of the section, and take axes GX and GY parallel to the edges of the section as indicated. Locating the centroid, we calculate, $\bar{y} = \frac{1}{3}$ in. and $\bar{x} = \frac{1}{4}$ in.

$$I_X = 17/192 \text{ in.}^4; \quad I_Y = 227/768 \text{ in.}^4; \quad I_{XY} = 35/384 \text{ in.}^4$$

The direction of the principal axes, GX' and GY' , is given by

$$\tan 2\theta = \frac{2 \times \frac{35}{384}}{\frac{227}{768} - \frac{17}{192}} = 0.88$$

and

$$2\theta = 41.33^\circ \quad \text{or} \quad 221.33^\circ$$

$$\theta = 20.67^\circ \quad \text{or} \quad 110.67^\circ$$

Having determined the values of 2θ , substitution of the known quantities into equations 4.12 and 4.13 will give us the principal moments of inertia. Thus,

$$I_{X'} = \frac{\frac{17}{192} + \frac{227}{768}}{2} - \sqrt{\left(\frac{\frac{227}{768} - \frac{17}{192}}{2}\right)^2 + \left(\frac{35}{384}\right)^2} = 0.054 \text{ in.}^4$$

$$I_{Y'} = 0.33 \text{ in.}^4$$

Determination of the principal axes may often be a laborious proceeding, unless an axis of symmetry exists. An axis of symmetry is always a principal axis and no calculation is then necessary to locate the principal axes. It is convenient to have available a formula, which will take into account the effects induced by bending about an axis other than a principal axis, and which will enable direct calculation of the resultant stresses to be made. Such a formula is derived in the following paragraph.

4.3. Stresses in Unsymmetrical Bending

Let bending moments M_x and M_y be applied to the section shown in Fig. 4.4, about axes XX and YY respectively, these axes passing through the centroid, G ,

of the section, but not being principal axes. We adopt the usual sign conventions for x and y , and a further convention that a positive bending moment produces compression in the upper right-hand quadrant.

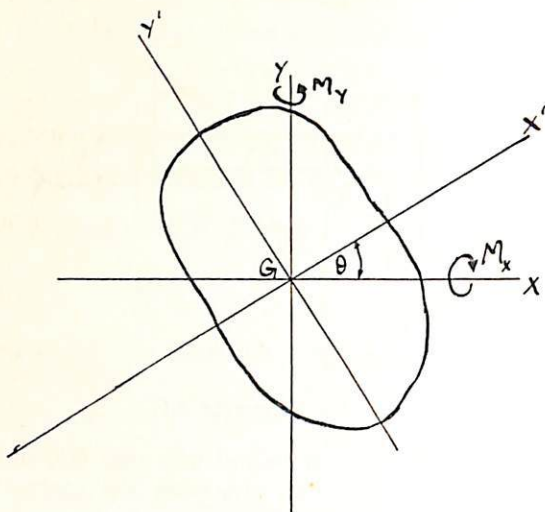


FIG. 4.4.—UNSYMMETRICAL BENDING.

As shown in para. 4.1, a bending moment M_x applied about an axis XX will produce a bending moment about YY of amount $\frac{M_x}{I_x} \cdot I_{xy}$. If M_x is positive, compression is produced in the upper right-hand quadrant, and the induced moment about YY , being supplied by the *internal* forces, will be *negative*. Similarly a positive moment applied about YY will induce a negative moment about XX .

Let M'_x and M'_y be the *effective* moments about XX and YY respectively, including all induced moments.

Then

$$M'_x = M_x - M_y \cdot \frac{I_{xy}}{I_y} \quad \text{Eq. 4.14}$$

and

$$M'_y = M_y - M_x \cdot \frac{I_{xy}}{I_x} \quad \text{Eq. 4.15}$$

From these two equations we obtain,

$$M'_x = M_x - \left(M_y - M_x \cdot \frac{I_{xy}}{I_x} \right) \frac{I_{xy}}{I_y}$$

i.e.,

$$M'_x = \frac{(M_x I_y - M_y I_{xy}) I_x}{I_x I_y - I_{xy}^2} \quad \text{Eq. 4.16}$$

Similarly,

$$M'_y = \frac{(M_y I_x - M_x I_{xy}) I_y}{I_x I_y - I_{xy}^2} \quad \text{Eq. 4.17}$$

The method outlined above approaches the problem by considering the physical aspects as represented by the induced and effective bending moments. Using the effective moments given by equations 4.16 and 4.17 to calculate stresses by

the simple bending formula we may add directly stresses due to bending about any two perpendicular axes, and we obtain, for a point whose co-ordinates are (x, y)

$$f = \frac{M'_x \cdot y}{I_x} + \frac{M'_y \cdot x}{I_y} \\ = \frac{(M_x I_y - M_y I_{xy})y}{I_x I_y - I_{xy}^2} + \frac{(M_y I_x - M_x I_{xy})x}{I_x I_y - I_{xy}^2} \quad \text{Eq. 4.18}$$

Equation 4.18 may be obtained also by resolving the applied moments into moments about the principal axes and adding the stresses due to these resolved moments, using the relationships obtained in para. 4.2, as follows. In Fig. 4.4 let GX' and GY' be the principal axes of the section, and let the angle $\angle XOX'$ be θ . Then θ , $I_{x'}$, and $I_{y'}$, satisfy the relationships derived in para. 4.2, and bending about either of the axes GX' and GY' does not induce bending about the other, so that stresses determined for bending about these axes separately may be added directly to give the resultant stresses. At a point whose co-ordinates are (x, y) referred to GX and GY , and (x', y') referred to GX' and GY' , the bending stress may therefore be written as

$$f = \frac{M_{x'} y'}{I_{x'}} + \frac{M_{y'} x'}{I_{y'}} \quad \text{Eq. 4.19}$$

$$\text{But} \quad M_{x'} = M_x \cos \theta - M_y \sin \theta \quad \text{Eq. 4.20}$$

$$\text{and} \quad M_{y'} = M_x \sin \theta + M_y \cos \theta \quad \text{Eq. 4.21}$$

Using these equations together with equations 4.5 and 4.6, we have

$$f = \frac{(M_x \cos \theta - M_y \sin \theta)(y \cos \theta - x \sin \theta)}{I_{x'}} \\ + \frac{(M_x \sin \theta + M_y \cos \theta)(x \cos \theta + y \sin \theta)}{I_{y'}} \\ = \frac{1}{I_x I_{y'}} \left[y \left\{ M_x \left(\frac{I_{x'} + I_{y'}}{2} + \cos 2\theta \frac{(I_{y'} - I_{x'})}{2} \right) + M_y \sin 2\theta \frac{(I_{x'} - I_{y'})}{2} \right\} \right. \\ \left. + x \left\{ M_x \frac{(I_{x'} - I_{y'})}{2} \sin 2\theta + M_y \left(\frac{I_{x'} + I_{y'}}{2} + \cos 2\theta \frac{(I_{x'} - I_{y'})}{2} \right) \right\} \right]$$

Now, from equations 4.12 and 4.13,

$$\frac{I_{x'} + I_{y'}}{2} = \frac{I_x + I_y}{2}$$

$$\frac{I_{x'} - I_{y'}}{2} = - \sqrt{\left(\frac{I_y - I_x}{2} \right)^2 + I_{xy}^2}$$

$$I_{x'} I_{y'} = I_x I_y - I_{xy}^2$$

Inserting these values, and those given for $\sin 2\theta$ and $\cos 2\theta$ in equation 4.11, into the expression for f above, we arrive finally at the same equation as 4.18.

Although the use of equation 4.18 will reduce the amount of work required to determine the bending stress at a given point, it will usually be necessary to locate the principal axes in order to ascertain the most highly stressed points.

The type of section illustrated in Fig. 4.3 is by no means the only type where consideration must be given to the problem of unsymmetrical bending. Propeller

blades and aeroplane wings are usually subjected to bending moments about axes other than principal axes, though in such cases the principal axes and the axes about which the bending moments are applied often so very nearly coincide that the corrections to the simple bending formula are neglected. It is generally desirable to check whether the axis about which bending is applied is a principal axis or not, and the necessary corrections should be made when calculating the bending stresses.

Example 4.2. Determine the stress at the point A in the section of Fig. 4.3, if the bending moments are $M_x = 1000$ lb. in., $M_y = 0$.

Using the quantities determined in Example 4.1, and applying equation 4.18, the stress at A is given by

$$f_A = \frac{-1000 \cdot \frac{227}{768} \cdot \frac{11}{12} + 1000 \cdot \frac{35}{384} \cdot \frac{17}{24}}{\frac{17}{192} \cdot \frac{227}{768} - \left(\frac{35}{384}\right)^2} \quad (x \text{ and } y \text{ are both negative at A})$$

$$= -11,550 \text{ lb. per sq. in.}$$

For comparison, the stress at A from the simple bending theory, neglecting unsymmetrical bending, would be given by

$$f_A = \frac{-1000 \times 11/12}{17/192} = -10,350 \text{ lb. per sq. in.}$$

It will be clear that in this case a considerable error would be introduced if the effects of unsymmetrical bending were neglected.

4.4. Curved Beams

The theory of simple bending as treated so far assumes the beam to be initially straight. There are cases, such as wheel forks, when the beam is curved before loading, and the assumptions made in deriving the simple bending formula for a straight beam do not yield the same results when applied to a curved beam. In the simple treatment of the curved beam, which follows, we still assume that plane cross-sections remain plane after bending. This assumption led to a linear variation of strain over the cross-section for the initially straight beam and to the neutral axis being located as passing through the centroid of the section. Neither of these results follows for the initially curved beam, although for convenience it is usual to locate points on the cross-section by their distances from the *centroidal* axis. The reason for the non-applicability of the straight beam formulæ is that over an elemental length between two plane cross-sections the fibres are not all initially of the same length in a curved beam. Since it is generally easier to locate the centroidal axis of the cross-section than the neutral axis for a curved beam, we shall take the centroidal axis as our reference line. We consider (Fig. 4.5) a small element of length of the beam with centroidal surface AB. The element is assumed small enough to be considered as an arc of a circle, the radius of curvature of AB being R , and the angle AOB being θ , with O the centre of curvature. This assumption implies also that θ is small. When this element of the beam is subjected to end couples M as indicated, let it deflect to the position shown dotted in the figure, the radius of curvature of A'B', the displaced centroidal surface, being R' and the angle subtended at O₁, the centre of curvature of A'B', being $(\theta + d\theta)$. We make the same assumptions as those made in developing the simple bending formulæ for straight beams in Volume I, with the following

addition :—in order to avoid unsymmetrical bending, we assume the cross-section to be symmetrical about the axis YY.

Let y denote the distance of a point in the section from the centroidal axis, and adopt the convention that y is positive when measured away from the centre

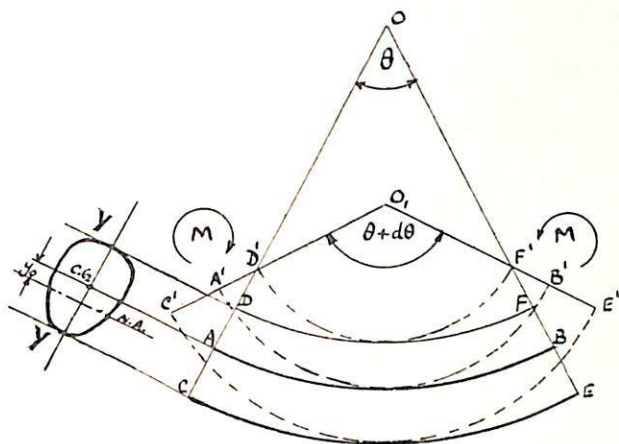


FIG. 4.5.—BENDING OF ELEMENT OF CURVED BEAM.

of curvature. Let the neutral axis, N.A., be at y_0 from the centroidal axis as indicated in the figure. It may be remarked here that the neutral axis is actually shown on the wrong side of the centroidal axis in the figure, but this is done intentionally in order to follow the principle of inserting all unknown quantities with positive signs. It will be found in practice that y_0 is generally negative.

The unstrained length of AB is $R\theta$ and the strained length, $A'B'$, is $R'(\theta + d\theta)$, so that the strain, ϵ_0 , of the centroidal fibre is

$$\epsilon_0 = \frac{R'(\theta + d\theta) - R\theta}{R\theta} = \frac{R'(\theta + d\theta)}{R\theta} - 1 \quad \text{Eq. 4.22}$$

The unstrained length of a fibre at distance y from the centroidal axis is $(R + y)\theta$ and its strained length is $(R' + y)(\theta + d\theta)$ so that the strain is

$$\begin{aligned} \epsilon &= \frac{(R' + y)(\theta + d\theta) - (R + y)\theta}{(R + y)\theta} \\ &= \frac{(R' + y)(\theta + d\theta)}{(R + y)\theta} - 1 \quad \text{Eq. 4.23} \end{aligned}$$

Substituting for $\frac{\theta + d\theta}{\theta}$ from equation 4.22 into equation 4.23, we obtain

$$\begin{aligned} \epsilon &= \frac{R' + y}{R + y} \cdot \frac{R}{R'} (1 + \epsilon_0) - 1 \\ &= \frac{Ry}{R + y} (1 + \epsilon_0) \frac{1}{R'} + \frac{R\epsilon_0 - y}{R + y} \\ &= \frac{Ry}{R + y} (1 + \epsilon_0) \frac{1}{R'} + \frac{(R + y)\epsilon_0 - y(1 + \epsilon_0)}{R + y} \end{aligned}$$

$$= \frac{Ry}{R+y}(1 + \epsilon_0) \left(\frac{1}{R'} - \frac{1}{R} \right) + \epsilon_0 \quad \text{Eq. 4.24}$$

We are assuming that the material obeys Hooke's law so that the stress at y from the centroidal axis is $f = E\epsilon$. The force on an element of area, dA , at y from the centroidal axis is then $f dA = E\epsilon dA$.

Since we assume the beam to be loaded by pure couples of amount M , the resultant longitudinal force on a cross-section is zero, and the total moment about the centroidal axis of the longitudinal forces on the elements of area of the section must be equal to M . Applying these two conditions gives

$$\int E\epsilon dA = ER(1 + \epsilon_0) \left(\frac{1}{R'} - \frac{1}{R} \right) \int \frac{y}{R+y} dA + E\epsilon_0 \int dA = 0 \quad \text{Eq. 4.25}$$

and

$$M = \int E\epsilon y dA = ER(1 + \epsilon_0) \left(\frac{1}{R'} - \frac{1}{R} \right) \int \frac{y^2}{R+y} dA + E\epsilon_0 \int y dA \quad \text{Eq. 4.26}$$

We note that the last term in equation 4.26 is zero, since y is measured from the centroidal axis. Then from equation 4.26 we have

$$ER(1 + \epsilon_0) \left(\frac{1}{R'} - \frac{1}{R} \right) = \frac{M}{\int \frac{y^2}{R+y} dA}$$

Putting this value in equation 4.25 and noting that the last term is $EA\epsilon_0$, where A is the area of the cross-section, gives

$$\epsilon_0 = - \frac{M \int \frac{y}{R+y} dA}{EA \int \frac{y^2}{R+y} dA} \quad \text{Eq. 4.27}$$

Substituting into equation 4.24 the expression for the strain at y from the centroidal axis becomes

$$\epsilon = \frac{y}{R+y} \left[\frac{M}{E \int \frac{y^2}{R+y} dA} \right] - \frac{M \int \frac{y}{R+y} dA}{EA \int \frac{y^2}{R+y} dA} \quad \text{Eq. 4.28}$$

The stress at y from the centroidal axis follows from this as

$$f = E\epsilon = \frac{M}{A} \left[\frac{- \int \frac{y}{R+y} dA}{\int \frac{y^2}{R+y} dA} + \frac{\frac{y}{R+y} A}{\int \frac{y^2}{R+y} dA} \right] \quad \text{Eq. 4.29}$$

Now

$$\int \frac{y}{R+y} dA = \int \left(1 - \frac{R}{R+y} \right) dA = A - R \int \frac{1}{R+y} dA \quad \text{Eq. 4.30}$$

and

$$\begin{aligned}\int \frac{y^2}{R+y} dA &= \int \left(y - R + \frac{R^2}{R+y} \right) dA = \int y dA - R \int dA + R^2 \int \frac{1}{R+y} dA \\ &= 0 - RA + R^2 \int \frac{1}{R+y} dA \quad \text{Eq. 4.31}\end{aligned}$$

Substituting from equations 4.30 and 4.31 into 4.29 we obtain

$$f = \frac{M}{A} \left[\frac{1}{R} + \frac{\frac{y}{R+y} \cdot A}{R \left(R \int \frac{1}{R+y} dA - A \right)} \right] = \frac{M}{AR} \left[1 + \frac{A}{Z} \cdot \frac{y}{R+y} \right] \quad \text{Eq. 4.32}$$

where

$$Z = R \int \frac{1}{R+y} dA - A \quad \text{Eq. 4.33}$$

The value of Z may be determined analytically for simple cross-sections, such as a circle or ellipse. For sections such that the integral in the expression for Z cannot be written in a form permitting of mathematical integration, a graphical construction may be adopted. A convenient graphical method for the determination of Z for any shape of cross-section is given in Appendix I.

4.5. Thin Curved Beams

In cases where the depth of a curved beam is very small, compared with the radius of curvature of the centroidal axis, the expressions just obtained may be simplified considerably. A typical example of this type of beam is a fuselage frame.

If y is small compared with R , the integral $\int \frac{y}{R+y} dA$ becomes simply $\int \frac{y}{R} dA$,

which is zero. It follows from equation 4.27 that ϵ_0 is zero. Further, $\int \frac{y^2}{R+y} dA$ becomes $\int \frac{y^2}{R} dA$, which we may write as $\frac{I}{R}$. It then follows from equation 4.26 that

$$\frac{1}{R'} - \frac{1}{R} = \frac{M}{EI} \quad \text{Eq. 4.34}$$

i.e., the change in curvature is equal to M/EI .

From equation 4.29 we obtain the stress at y from the centroidal axis as

$$f = \frac{M}{A} \left[0 + \frac{Ay/R}{I/R} \right] = \frac{My}{I} \quad \text{Eq. 4.35}$$

Equations 4.34 and 4.35 indicate that for a curved beam of this type, in which the section depth is small compared with the radius of curvature, the longitudinal stresses due to bending may be determined with reasonable accuracy by application of the simple bending theory for straight beams. The analysis of a fuselage ring will not be considered in this text, the general problem being somewhat complex.

DISTRIBUTION OF SHEAR STRESS IN BEAMS

5.1. General

No attempt has been made in earlier chapters to consider any distribution of shear stress other than uniform—and in such cases as the shear of a bolt or rivet it is customary to assume uniformity of shear stress over the cross-section. In a member which transmits shear by bending, the distribution of shear stress is, in general, not uniform over a cross-section, and we shall discuss this problem in the succeeding paragraphs.

5.2. Shear Stress Distribution in a Solid Beam

Let AB and CD be two cross-sections of a uniform beam, δx apart, and assume that M and $M + \delta M$ are the bending moments at AB and CD respectively. Consider the equilibrium of the element ACFE (EF being at distance y_1 from

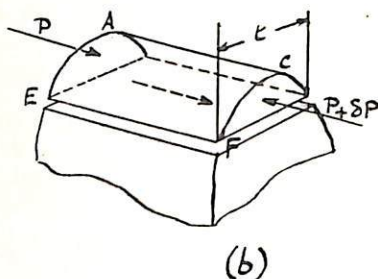
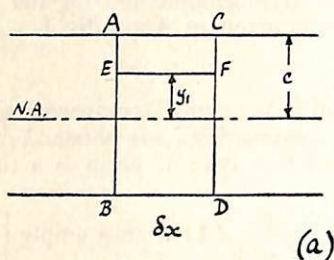


FIG. 5.1.—LONGITUDINAL SHEAR IN A BEAM.

the neutral axis), with reference to forces in the longitudinal direction. We know from the theory of simple bending that the longitudinal stress, f , on an element of the cross-section at distance y from the neutral axis is given by $f = \frac{My}{I}$, where I is the second moment of area of the section about the neutral axis. Hence the longitudinal force on such an element, of area δA , will be $f \cdot \delta A = \frac{My}{I} \cdot \delta A$. It follows that the longitudinal force on the portion AE of the cross-section AB is

$$\int_{y_1}^c \frac{MydA}{I} = \frac{M}{I} \int_{y_1}^c ydA$$

Then, referring to Fig. 5.1, we see that

$$(P + \delta P) - P = \frac{M + \delta M}{I} \int_{y_1}^c ydA - \frac{M}{I} \int_{y_1}^c ydA = \frac{\delta M}{I} \int_{y_1}^c ydA$$

i.e.,

$$\delta P = \frac{\delta M}{I} \int_{y_1}^c ydA$$

For equilibrium of the beam element ACFE, a force F must be applied along the face EF (Fig. 5.1 (b)) such that $F = \delta P$.

This force F is a shear force on the face EF, and with our definition of shear stress as shear force divided by the area on which the shear force acts, we may write, if we specify that the shear stress is distributed uniformly across the width of the section,

$$\text{Shear stress on EF} = q_1 = \frac{F}{t \cdot \delta x} = \frac{\delta P}{t \cdot \delta x} = \frac{\delta M}{t \cdot \delta x \cdot I} \int_y^c y dA$$

As δx tends to zero, this becomes

$$q_1 = \frac{\frac{dM}{dx}}{It} \int_{y_1}^c y dA$$

From the relationship between shear force and bending moment derived in an earlier chapter (I.10.7), we know that $\frac{dM}{dx} = S$, where S is the shear force. Hence,

$$q_1 = \frac{S}{It} \int_{y_1}^c y dA \quad \dots \dots \dots \text{Eq. 5.1}$$

The quantity $\int_{y_1}^c y dA$ is the first moment about the neutral axis of the area of the portion AE of the cross-section, and may be denoted by the symbol Q_1 , where the suffix 1 indicates that the part of the cross-section lying beyond $y = y_1$ is to be considered. We may then write equation 5.1 in the form

$$q_1 = \frac{SQ_1}{It} \quad \dots \dots \dots \text{Eq. 5.2}$$

From para. 3.2 we know that a longitudinal shear stress q_1 will be accompanied by a shear stress of equal magnitude in a perpendicular direction—i.e., over the cross-section of the beam. Hence equation 5.2 gives the value of the shear stress at a point on the cross-section at distance y_1 from the neutral axis.

It must be emphasised that in the above analysis it is assumed that the shear stress is uniform over the width of the section.

5.3. Distribution of Shear Stress for Certain Cross-sections

Equation 5.2 will now be applied to some cross-sections of simple geometric shape in order to see how much error is involved if the assumption of uniform shear stress is made.

(i) *Rectangle.* From Fig. 5.2,

$$\begin{aligned} Q_1 &= \text{Moment of shaded area about N.A.} \\ &= B \left(\frac{D}{2} - y_1 \right) \left\{ y_1 + \frac{1}{2} \left(\frac{D}{2} - y_1 \right) \right\} \\ &= \frac{B}{2} \left\{ \left(\frac{D}{2} \right)^2 - y_1^2 \right\} \end{aligned}$$

Then equation 5.2 gives for the shear stress at y_1 from the N.A.,

$$q_1 = \frac{\frac{SB}{2} \left\{ \left(\frac{D}{2} \right)^2 - y_1^2 \right\}}{\frac{BD^3}{12} B}$$

$$= \frac{6S \left\{ \left(\frac{D}{2} \right)^2 - y_1^2 \right\}}{BD^3}$$

If now we give y_1 values ranging from $-D/2$ to $+D/2$ we obtain the value of the shear stress at all points on the cross-section and we can plot the curve

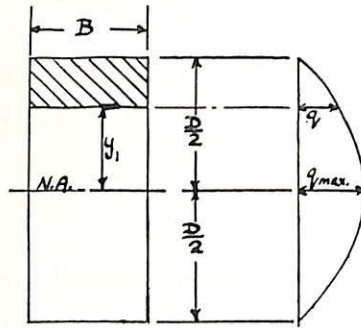


FIG. 5.2.—DISTRIBUTION OF SHEAR STRESS OVER A RECTANGULAR SECTION.

of distribution shown in Fig. 5.2. In this case the curve is parabolic. The maximum value of q occurs at the neutral axis, where $y = 0$, and is given by

$$q_{\max.} = \frac{6S}{BD^3} \cdot \left(\frac{D}{2} \right)^2 = \frac{3}{2} \cdot \frac{S}{BD} \quad \text{Eq. 5.3}$$

$$q_{\max.} = \frac{3}{2} \times \text{mean shear stress} \quad \text{Eq. 5.4}$$

where the mean shear stress is the stress calculated on the assumption of uniform stress—i.e., shear force divided by cross-sectional area.

It should be noted that the assumption of uniform stress takes no account of the physical fact that the shear stress must be zero at the free edges at the top and bottom of the section, whilst equation 5.2 does give this result.

(ii) *Circle*. By a similar method it may be shown that the maximum shear stress (at the neutral axis) for a circular section is $4/3$ times the mean stress. The proof of this is left as an example to be worked by the student.

(iii) *I-Section*. In this case, direct application of equation 5.2 will give the distribution of shear stress over the depth of the web, but will not apply in the same way to the flanges. It will not give the distribution of shear stress over the *depth* of a flange, but it can be used to determine the approximate distribution over the *width* of a flange, particularly if the depth of the flange is small compared with that of the web.

Initially, the symmetrical section in Fig. 5.3 will be considered under the assumption that equation 5.2 may be applied over the full depth. The modifica-

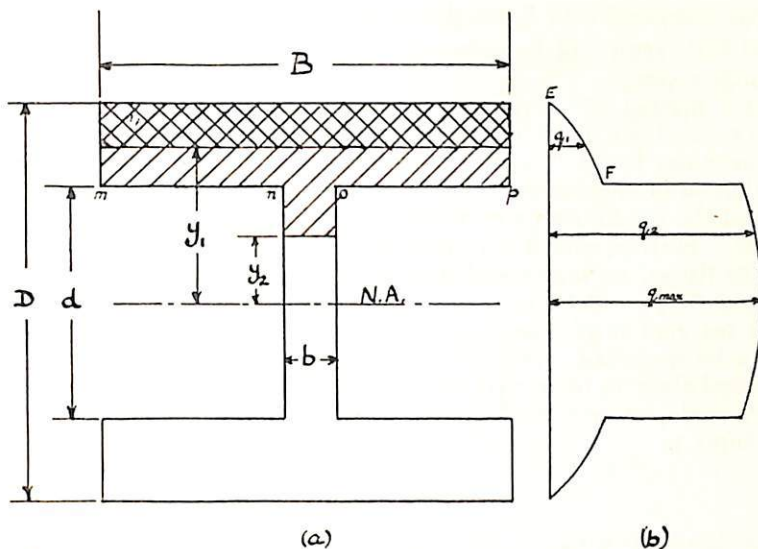


FIG. 5.3.—DISTRIBUTION OF SHEAR STRESS OVER AN I-SECTION, FROM APPLICATION OF EQUATION 5.2.

tions necessary in connection with the flange will then be discussed in an elementary manner.

Applying equation 5.2 for $y = y_1$ (Fig. 5.3) gives

$$\begin{aligned} q_1 &= \frac{SB \left(\frac{D}{2} - y_1 \right) \left(\frac{D}{2} - \frac{D/2 - y_1}{2} \right)}{IB} \\ &= \frac{S \left\{ \left(\frac{D}{2} \right)^2 - y_1^2 \right\}}{2I} \quad \dots \dots \dots \text{Eq. 5.5} \end{aligned}$$

For $y = y_2$ (Fig. 5.3), we have

$$\begin{aligned} q_2 &= \frac{S}{Ib} \left\{ \frac{B}{2} \left[\left(\frac{D}{2} \right)^2 - \left(\frac{d}{2} \right)^2 \right] + b \left(\frac{d}{2} - y_2 \right) \left(\frac{d}{2} + y_2 \right) \right\} \\ &= \frac{S}{2Ib} \left\{ B \left[\left(\frac{D}{2} \right)^2 - \left(\frac{d}{2} \right)^2 \right] + b \left[\left(\frac{d}{2} \right)^2 - y_2^2 \right] \right\} \quad \dots \dots \dots \text{Eq. 5.6} \end{aligned}$$

Plotting q_1 and q_2 against the depth yields the distribution curve shown in Fig. 5.3 (b).

The minimum value of q_2 occurs when $y_2 = \frac{d}{2}$, and is

$$q_{2\min.} = \frac{S}{2I} \cdot \frac{B}{b} \left[\left(\frac{D}{2} \right)^2 - \left(\frac{d}{2} \right)^2 \right]$$

Similarly, when $y_2 = 0$,

$$q_{2\max.} = \frac{S}{2Ib} \left\{ B \left[\left(\frac{D}{2} \right)^2 - \left(\frac{d}{2} \right)^2 \right] + b \left(\frac{d}{2} \right)^2 \right\}$$

If b is small compared with B , the difference between $q_{2_{\min}}$ and $q_{2_{\max}}$ will be very small, and little error will be involved in assuming uniform shear stress in the web in such a case.

That the portion EF of the distribution curve is incorrect becomes obvious when it is realised that along the faces mn and op, (Fig. 5.3 (a)), there are free edges, so that there can be no "vertical" shear stress at these edges. Consequently the distribution of shear stress cannot be uniform over the width, mnop, of the section, and the conditions under which equation 5.2 was developed are not met in this case. Further, since the vertical shear stress is zero at the outer and inner faces of the flange, we may reasonably assume that the maximum vertical shear stress in the flange will be very small as d approaches D —i.e., as the depth of the flange becomes small compared with the depth of the section—and may in most cases be neglected. In such circumstances we should assume the whole of the vertical shear to be carried by the web, and if also b were small compared with B we could, as mentioned above, assume uniform shear stress in the web, leading simply to

$$q_{\text{web}} = \frac{S}{bd} \quad \text{Eq. 5.7}$$

Further justification for this assumption may be obtained by determining from equation 5.6 the total shear force carried by the web, as follows. For an element of depth dy at distance y_2 from the neutral axis the shear force will be $q_2 \cdot b \cdot dy$, and the shear force in the web will be determined by integrating this expression between the limits $y = -d/2$ and $y = d/2$. Hence the shear in the web, S_w , is

$$\begin{aligned} S_w &= \int_{-\frac{d}{2}}^{\frac{d}{2}} q_2 b dy = 2 \int_0^{\frac{d}{2}} q_2 b dy \\ &= \frac{2S}{2I} \int_0^{\frac{d}{2}} \left\{ B \left[\left(\frac{D}{2} \right)^2 - \left(\frac{d}{2} \right)^2 \right] + b \left[\left(\frac{d}{2} \right)^2 - y^2 \right] \right\} dy \\ &= \frac{S}{I} \left\{ \frac{B}{8} (dD^2 - d^3) + \frac{bd^3}{12} \right\} \end{aligned}$$

For the section considered,

$$\begin{aligned} I &= \frac{BD^3}{12} - \frac{(B-b)d^3}{12} = \frac{B}{12} (D^3 - d^3) + \frac{bd^3}{12} \\ \therefore \frac{S_w}{S} &= \frac{\frac{B}{8} (dD^2 - d^3) + \frac{bd^3}{12}}{\frac{B}{12} (D^3 - d^3) + \frac{bd^3}{12}} \end{aligned}$$

As d approaches D , both $(dD^2 - d^3)$ and $(D^3 - d^3)$ tend to zero, and the ratio S_w/S approaches unity. Hence, when the flange depth is small compared with the depth of the section, we are justified in assuming the whole vertical shear to be carried in the web.

It should be noted that the above does not imply zero shear stress in the flanges—it implies only zero *vertical* shear stress. There will be a longitudinal shear stress, accompanied by a *horizontal* shear stress, and this will be discussed briefly in the next paragraph.

5.4. Shear Stress in the Flanges of an I-Section

The development of the necessary formulæ follows the lines of that in para. 5.2, but we now consider the equilibrium of the elementary block shown shaded in Fig. 5.4. We assume that the flange depth is small compared with the section

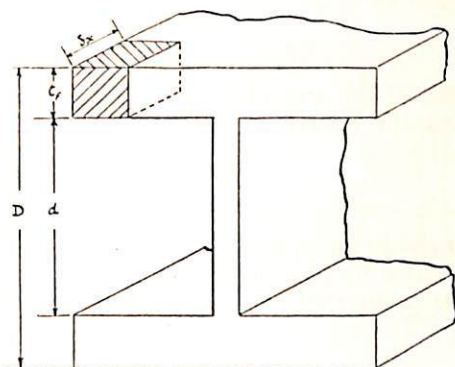


FIG. 5.4.—ELEMENT OF FLANGE CONSIDERED IN DETERMINATION OF SHEAR STRESS DISTRIBUTION.

depth, which will generally be the case for sections used in aircraft structures. We can then say that the longitudinal stress due to bending may be assumed constant over the flange depth. This is equivalent to saying that $\frac{M \cdot D/2}{I}$ and $\frac{M \cdot d/2}{I}$ may be assumed equal if $d/2$ is very nearly equal to $D/2$. Let the area of the face on which P acts be δA , and let h be the distance of the centroid of δA above the neutral axis.

Then

$$\begin{aligned}\delta P &= (P + \delta P) - P \\ &= \left\{ \frac{(M + \delta M)h}{I} - \frac{Mh}{I} \right\} \delta A \\ &= \frac{\delta M}{I} \cdot h \delta A\end{aligned}$$

This force, δP , must be balanced by a shear force, F , as indicated in Fig. 5.5 (a), and the assumption we have made above indicates that this shear is uniformly

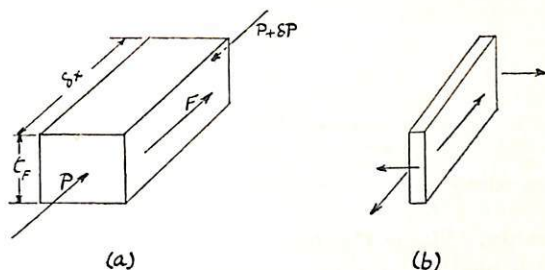


FIG. 5.5.—EQUILIBRIUM OF ELEMENT OF FLANGE OF AN I-SECTION.

distributed over the *depth* of the flange at the section considered. Then if t_F be the flange thickness, the shear stress in the longitudinal direction, q_F , is given by

$$q_F = \frac{F}{t_F \delta x} = \frac{\delta M \cdot h \delta A}{t_F \delta x \cdot I}$$

As δx tends to zero, this becomes

$$q_F = \frac{S \cdot h \delta A}{I t_F}$$

$h \delta A$ is the first moment of the area of flange, between the outer edge and the point considered, about the neutral axis, and may be denoted again by Q_F . Hence,

$$q_F = \frac{S \cdot Q_F}{I t_F} \quad \dots \dots \dots \text{Eq. 5.8}$$

This shear stress gives rise to an equal shear stress at right-angles, and reference to Fig. 5.5 (b) will show that for equilibrium of an element of the flange the complementary shear stress is directed along the width of the flange.

5.5. Shear Flow

In the preceding paragraphs we have referred throughout to shear force and shear stress. It is often more convenient, particularly when dealing with thin metal structures such as are found in aircraft, to work in terms of *shear per unit length*, which is equal to the shear stress multiplied by the thickness, the length being measured in the direction of the shear stress. For example, in the case of the I-section just discussed the direction of the shear stress in the web is vertical, the thickness of the web is b , and the shear per unit length is then $q_w \cdot b$. Similarly the shear per unit length in the flange is $q_F \cdot t_F$. A more convenient term for shear per unit length is *shear flow*. In this book we have used q to denote shear stress, and we shall generally denote shear flow by s , so that we may write,

$$s = qt \quad \text{or} \quad q = s/t$$

For the I-section discussed previously,

$$s_w = S/d \quad \text{and} \quad s_F = S Q_F / I$$

The direction of the shear flow in the I-section will be as indicated in Fig. 5.6. s_F increases linearly from zero at the outer edge, and s_w is constant.

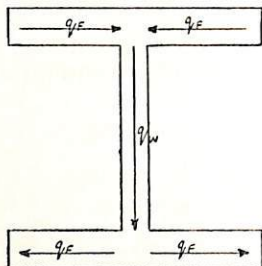


FIG. 5.6.—DIRECTION OF SHEAR STRESS (AND SHEAR FLOW) IN AN I-SECTION.

5.6. Channel Section. Shear Centre

In the case of a channel section, consideration of the foregoing discussion indicates a shear flow as shown in Fig. 5.7 as the internal reaction to the applied

shear force S . It is apparent that the shear forces in the upper and lower flanges together constitute a couple, so that unless the applied shear force S is located at a certain distance from the channel web, as indicated, to provide a balancing couple formed by S and the shear force in the web, the section will twist. The

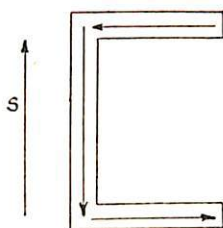


FIG. 5.7.—SHEAR FLOW IN CHANNEL SECTION.

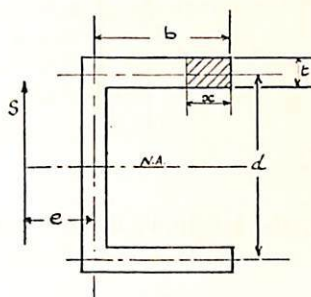


FIG. 5.8.—LOCATION OF SHEAR CENTRE OF CHANNEL SECTION.

point at which S must be applied to avoid twisting is the *shear centre* of the section (or sometimes *flexural centre*). The student is warned against the indiscriminate use of these two terms as meaning the same point. More precise definitions of shear centre and flexural centre, with particular reference to present-day usages, are given in Appendix III.

The shear centre of the channel section of Fig. 5.8 may be located as follows:—

At a point distant x from the edge of a flange the shear flow is $s_F = \frac{S}{I}(xt)\frac{d}{2}$, so that the total shear force in the flange is

$$S_F = \int_0^b s_F dx = \frac{Std^2}{4I}$$

The couple formed by the shear forces in the two flanges is $S_F \cdot d$, and the distance, e , of the shear centre from the web centre-line is given by

$$S \cdot e = \frac{Std^2b^2}{4I}$$

Hence,

$$e = \frac{td^2b^2}{4I}$$

5.7. Thin-Web Beams

Many aircraft structures contain members subjected to bending which comprise, as essential elements, longitudinal flange members and thin shear webs. An idealised version of such a member is shown in Fig. 5.9. In a beam of this type, where the web thickness is very small, it is usual to assume the bending moment to be met by end loads in the flanges only, the web contributing nothing to the moment of resistance. Further, the longitudinal stress in the flange due to bending is assumed to be distributed uniformly over the cross-section of the flange, so that if the flange cross-sectional area is A_F and the bending moment

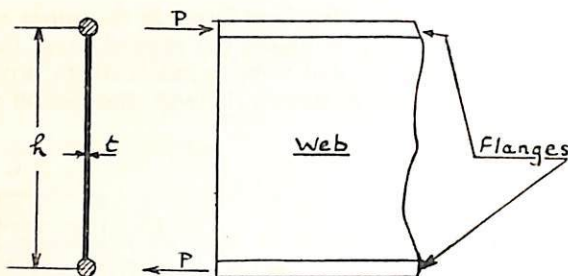


FIG. 5.9.—IDEALISED THIN-WEB BEAM.

is M , the longitudinal stress f is given by

$$f = \frac{P}{A_F} = \frac{M}{hA_F} \quad \text{Eq. 5.9}$$

The theory of simple bending gives

$$f = \frac{M \cdot h/2}{I}$$

Comparing this with equation 5.9 indicates that we can say that in equation 5.9 we are using an effective value for I equal to $\frac{A_F \cdot h^2}{2} (= 2A_F \cdot \left\{\frac{h}{2}\right\}^2)$. For the symmetrical section considered, this quantity is $I_{N.A.}$ as calculated if we neglect I_G for the flanges and $I_{N.A.}$ for the web. If the two flanges are not the same, we may still neglect the bending strength of the web and calculate $I = \Sigma A_F \cdot y^2$, where A_F is the cross-sectional area of a flange whose centroid is at y from the neutral axis. Neglecting the bending strength of the web implies also the assumption of uniform shear flow in the web, and in general we go one stage further and assume the whole shear to be carried by the web, as for the I-section.

With these assumptions we consider a length Δx of the beam, over which we may assume S to be constant (Fig. 5.10). If we isolate the upper flange, as in Fig. 5.11 (a), we note that it will be in equilibrium under the forces P_A , P_B

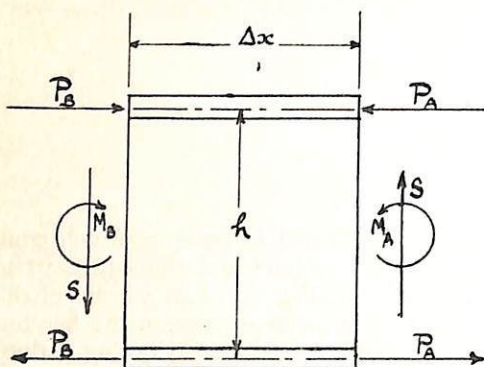


FIG. 5.10.—FORCES AND MOMENTS ON AN ELEMENTAL LENGTH OF A THIN-WEB BEAM.

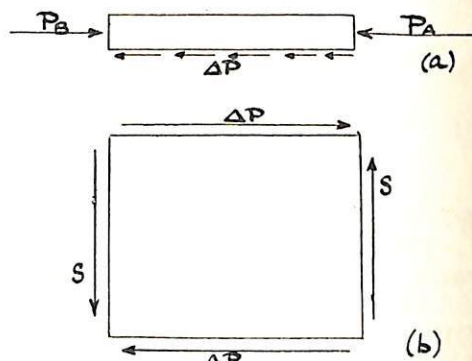


FIG. 5.11.—EQUILIBRIUM OF ELEMENTS OF THIN-WEB BEAM. (a) FLANGE; (b) WEB.

and the distributed force ΔP , which is applied to the flange by the web. Consequently, we may write,

$$\Delta P = P_B - P_A = \frac{(M_B - M_A)}{h} \quad \text{Eq. 5.10}$$

The forces acting on the web, if this is similarly isolated, are as shown in Fig. 5.11 (b). The equilibrium of the beam element as a whole indicates that $M_B - M_A = S \cdot \Delta x$, and the combination of this relationship with equation 5.10 shows that the system of forces in Fig. 5.11 (b) is in equilibrium. It follows then that

$$h \cdot \Delta P = S \cdot \Delta x$$

or
$$\frac{\Delta P}{\Delta x} = \frac{S}{h} = \text{shear flow in web} = s \quad \text{Eq. 5.11}$$

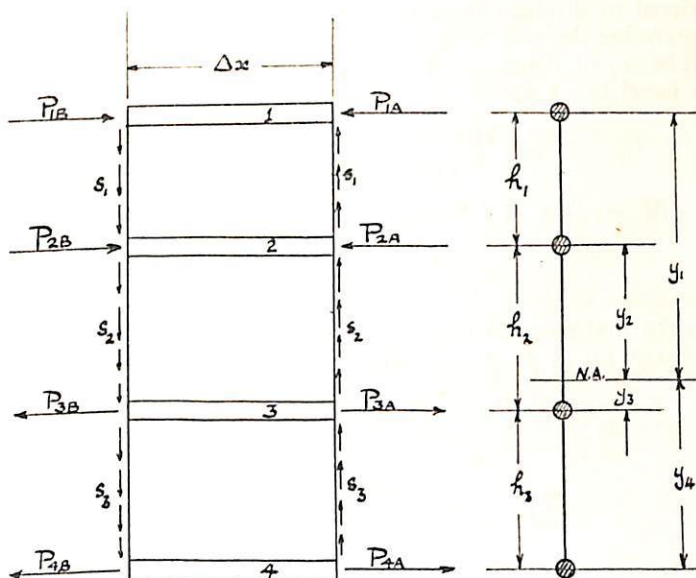


FIG. 5.12.—IDEALISED FOUR-FLANGE THIN-WEB BEAM.

Hence, the shear flow in the web is equal to the rate of change of end load in the flange. If the shear flow in the web of a beam is thus regarded as arising from the variation of flange end load, it is considered that a clearer picture is obtained of the mechanism of shear transference along a beam. Furthermore, no matter how the cross-section of a beam varies along the length of the beam, the shear flow in the web must always be related in this way to the rate of change of flange end load along the span.

The above analysis will now be extended to cover the multiflange type of thin-web beam. Fig. 5.12 shows an idealised four-flange beam. As before, bending is assumed to be resisted by end loads in the flanges only, and the shear to be carried entirely by the web elements. The position of the neutral axis will be such that this axis passes through the centroid of the flange areas—i.e., if A_F denotes the area of a flange at distance y from the neutral axis, then $\sum A_F \cdot y = 0$. As suggested above, we calculate an effective I as equal to $\sum A_F \cdot y^2$, and apply

the theory of simple bending to give

$$P_{1A} = A_{F_1} \cdot \frac{M_A y_1}{\sum A_F y^2} = \frac{M_A}{I} \cdot A_{F_1} y_1$$

$$P_{2A} = \frac{M_A}{I} \cdot A_{F_2} y_2$$

$$P_{3A} = \frac{M_A}{I} \cdot A_{F_3} y_3$$

$$P_{4A} = \frac{M_A}{I} \cdot A_{F_4} y_4$$

(It may be noted that these relationships follow from the assumption that plane sections remain plane so that strains, and hence stresses within the elastic limit, are proportional to distance from the neutral axis.)

If we determine the end loads at station B in a similar manner, consideration of the equilibrium of flange no. 1 leads to the determination of the shear flow in the web panel no. 1 as

$$s_1 = \frac{P_{1B} - P_{1A}}{\Delta x} = \frac{(M_B - M_A)}{I \cdot \Delta x} A_{F_1} y_1$$

Since $M_B - M_A = S \Delta x$, it follows that

$$s_1 = \frac{S Q_1}{I} \quad \dots \quad \text{Eq. 5.12}$$

where Q_1 is the first moment of area of flange no. 1 about the N.A. The equilibrium of flange no. 2 gives, with the notation of Fig. 5.13,

$$F = s_1 \Delta x + P_{2B} - P_{2A} = s_1 \Delta x + \frac{(M_B - M_A)}{I} A_{F_2} y_2$$

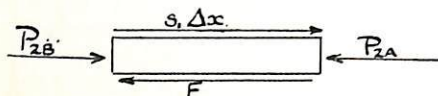


FIG. 5.13.—EQUILIBRIUM OF FLANGE NO. 2 OF BEAM OF FIG. 5.12.

so that

$$s_2 = \frac{F}{\Delta x} = s_1 + \frac{S Q_2}{I} = \frac{S}{I} (Q_1 + Q_2) \quad \dots \quad \text{Eq. 5.13}$$

It will be observed that this equation is of the same form as equation 5.2, if we remember that shear flow is shear stress times thickness and if we specify that Q is calculated for the material assumed effective in bending only.

If we write, for a particular flange, $\Delta P = P_B - P_A$, the relationship between s_1 and s_2 becomes

$$s_2 = s_1 + \frac{\Delta P_2}{\Delta x} \quad \dots \quad \text{Eq. 5.14}$$

This form is convenient to remember, and, of course, leads to the limiting expression for the shear flow at a given station along the beam,

$$s_2 = s_1 + \frac{dP_2}{dx} \quad \dots \quad \text{Eq. 5.15}$$

If we continue this procedure, we obtain the general relationship

$$s_n = s_{n-1} + \frac{dP_n}{dx} \quad \text{Eq. 5.16 (a)}$$

or, if the shear flow is constant over a length, Δx , of the beam,

$$s_n = s_{n-1} + \frac{\Delta P_n}{\Delta x} \quad \text{Eq. 5.16 (b)}$$

Taking due account of the direction of ΔP , it may be verified that the total shear force in the web given by application of equation 5.13 is equal to the applied shear force, S . In the case of the section in Fig. 5.12, the total shear force in the web, S_w , is given by

$$\begin{aligned} S_w &= s_1 h_1 + s_2 h_2 + s_3 h_3 \\ &= \frac{S}{I} \{h_1 Q_1 + h_2(Q_1 + Q_2) + h_3(Q_1 + Q_2 - Q_3)\} \end{aligned}$$

(The negative sign appears with Q_3 to take account of the direction of ΔP_3 .) Inserting the values of Q_1 , Q_2 and Q_3 we have

$$\begin{aligned} S_w &= \frac{S}{I} \{A_{F_1} y_1 (h_1 + h_2 + h_3) + A_{F_2} y_2 (h_2 + h_3) - A_{F_3} y_3 h_3\} \\ &= \frac{S}{I} \{A_{F_1} y_1 (y_1 + y_4) + A_{F_2} y_2 (y_2 + y_4) - A_{F_3} y_3 (y_4 - y_3)\} \\ &= \frac{S}{I} \{A_{F_1} y_1^2 + A_{F_2} y_2^2 + A_{F_3} y_3^2 + y_4 (A_{F_1} y_1 + A_{F_2} y_2 - A_{F_3} y_3)\} \end{aligned}$$

But $A_{F_1} y_1 + A_{F_2} y_2 - A_{F_3} y_3 - A_{F_4} y_4 = 0$, since the y 's are measured from the neutral axis. Hence $A_{F_1} y_1 + A_{F_2} y_2 - A_{F_3} y_3 = A_{F_4} y_4$, and

$$S_w = \frac{S}{I} \{\Sigma A_F y^2\} = \frac{S \cdot I}{I} = S$$

The general formula derived above (equation 5.16) enables us to determine the shear flow in all web elements, if we can start from an element where the shear flow is known. In the section just considered, we start with the knowledge that the shear flow on the outer (free) edge of flange no. 1 is zero. Having calculated the value of ΔP for each flange from the simple bending formula we can proceed to the evaluation of the shear flow in each web element. For a closed section such as is shown in Fig. 5.14 there is no free edge from which we can start, but continued application of equation 5.16 will give the shear flows for all web panels in terms of one of them, say s_1 . The final determination of s_1 then depends on the equilibrium of the internal shear flow with the external forces in the plane of the section. The analysis of such sections will be treated in a subsequent chapter. The procedure for a section of the type shown in Fig. 5.12 is illustrated by the following example.

Example 5.1. Fig. 5.15 shows a portion of a skin-stringer panel which acts as a beam. Assuming the bending to be met entirely by end loads in the flanges, the web carrying all the shear, determine the shear stress in each panel, if the shear force and bending moment at A are 300 lb. and 4000 lb. in. respectively in directions as indicated in the figure. The flange centroids may be assumed

coincident with the rivet lines, and the shear force may be assumed constant between A and B.

Let the neutral axis of the section be at distance $\bar{y}$ from the bottom rivet line.

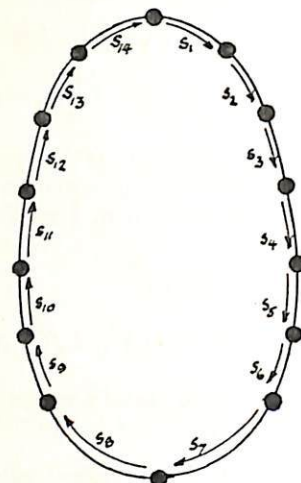


FIG. 5.14.—SHEAR FLOW IN CLOSED SECTION.

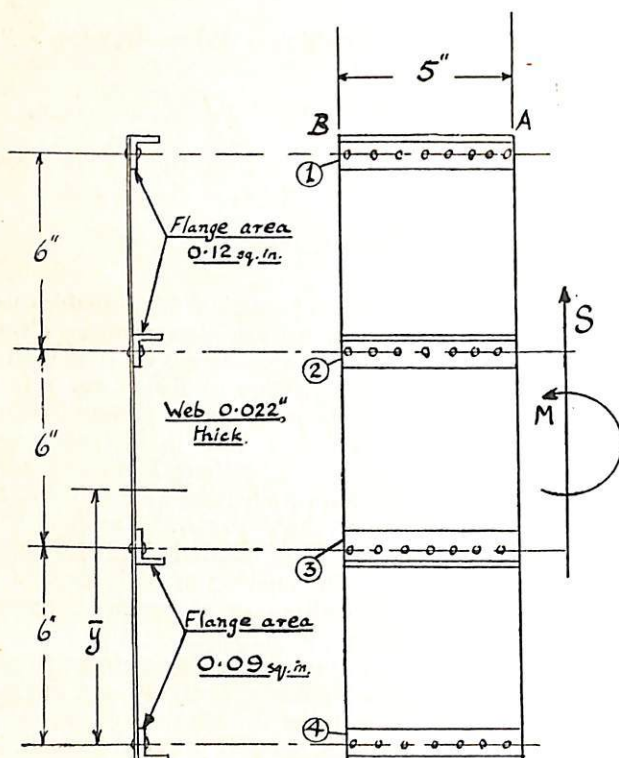


FIG. 5.15.—BEAM FOR EXAMPLE 5.1.

Since we assume that only the flanges are effective in bending, the neutral axis passes through the centroid of the flange areas, and we have

$$\bar{y} = \frac{0.12(12 + 18) + 0.09 \times 6}{2(0.12 + 0.09)} = 9.86 \text{ in.}$$

The second moment of area about the neutral axis, I , is given by

$$\begin{aligned} I = \Sigma A_F y^2 &= 0.12(2 \cdot 14^2 + 8 \cdot 14^2) + 0.09(3.86^2 + 9.86^2) \\ &= 0.55 + 7.95 + 1.34 + 8.75 \\ &= 18.59 \text{ in.}^4 \end{aligned}$$

Numbering the flanges from 1 to 4, from top to bottom, and denoting the end load in a flange by P , we have at station A,

$$P_{1A} = \frac{4000 \times 8.14 \times 0.12}{18.59} = 210 \text{ lb.}$$

$$P_{2A} = \frac{4000 \times 2.14 \times 0.12}{18.59} = 55.2 \text{ lb.}$$

$$P_{3A} = \frac{4000 \times (-3.86) \times 0.09}{18.59} = -74.7 \text{ lb.}$$

$$P_{4A} = \frac{4000 \times (-9.86) \times 0.09}{18.59} = -191 \text{ lb.}$$

At station B the bending moment is $4000 + 300 \times 5 = 5500 \text{ lb. in.}$, so that we obtain by similar calculations:—

$$P_{1B} = 289 \text{ lb.}, \quad P_{2B} = 76 \text{ lb.}, \quad P_{3B} = -102.8 \text{ lb.}, \quad P_{4B} = -262.5 \text{ lb.}$$

Then,

$$\Delta P_1 = 289 - 210 = 79 \text{ lb.}$$

$$\Delta P_2 = 76 - 55.2 = 20.8 \text{ lb.}$$

$$\Delta P_3 = -102.8 + 74.7 = -28.1 \text{ lb.}$$

$$\Delta P_4 = -262.5 + 191 = -71.5 \text{ lb.}$$

Applying equation 5.16 (b), we determine the shear flow in each web panel as follows,

$$s_{12} = \frac{\Delta P_1}{5} = 15.8 \text{ lb. per in.}$$

$$s_{23} = s_{12} + \frac{\Delta P_2}{5} = 15.8 + 4.16 = 19.96 \text{ lb. per in.}$$

$$s_{34} = s_{23} + \frac{\Delta P_3}{5} = 19.96 - 5.62 = 14.34 \text{ lb. per in.}$$

The shear stress in each panel is obtained by dividing the shear flow by the web thickness, so that

$$q_{12} = \frac{s_{12}}{0.022} = 718 \text{ lb. per sq. in.}$$

$$q_{23} = \frac{s_{23}}{0.022} = 907 \text{ lb. per sq. in.}$$

$$q_{34} = \frac{s_{34}}{0.022} = 652 \text{ lb. per sq. in.}$$

The above solution illustrates the detailed procedure, but the work may be reduced considerably by applying equations of the type 5.13 directly as follows :—

$Q_1 = \text{moment of area of flange no. 1 about N.A.} = 0.12 \times 8.14 = 0.977 \text{ in.}^3$

$Q_2 = 0.257 \text{ in.}^3$ $Q_3 = -0.347 \text{ in.}^3$ $Q_4 = -0.887 \text{ in.}^3$

Then $s_{12} = \frac{300 \times 0.977}{18.59} = 15.8 \text{ lb. per in.}$

$s_{23} = \frac{300(0.977 + 0.257)}{18.59} = 19.92 \text{ lb. per in.}$

$s_{34} = \frac{300(0.977 + 0.257 - 0.347)}{18.59} = 14.32 \text{ lb. per in.}$

These results agree within the limits of slide rule error with those obtained above, and the calculations have been greatly lessened.

5.8. Tapered Beams—Shear carried by Flanges

In the preceding paragraphs it has been assumed that the flanges of the beam were parallel, but this is not always the case. Consider the beam in Fig. 5.16,

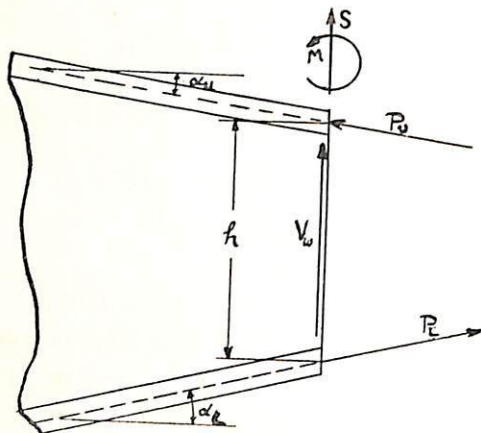


FIG. 5.16.

the upper and lower flanges of which are inclined at angles α_U and α_L respectively to the horizontal axis of the beam. Let the end loads in the flanges at a section, where the shear force is S and the bending moment is M , be P_U and P_L as shown. These end loads, together with the shear in the web, provide a system of forces equivalent to S and M . The forces P_U and P_L may be resolved into horizontal and vertical components. Denoting horizontal components by H and vertical components by V , and letting the distance between the centroids of the flanges be h , we have, from the conditions of equilibrium,

$$H_U = H_L$$

$$H_U \cdot h = H_L \cdot h = M$$

$$V_U + V_L + V_w = S$$

where V_w is the shear in the web.

From the geometry of the system we have also,

$$H_U = P_U \cos \alpha_U; \quad H_L = P_L \cos \alpha_L; \quad V_U = P_U \sin \alpha_U; \quad V_L = P_L \sin \alpha_L$$

It follows from the above relationships that

$$\left. \begin{aligned} P_U &= H_U \sec \alpha_U = \frac{M}{h} \sec \alpha_U \\ P_L &= H_L \sec \alpha_L = \frac{M}{h} \sec \alpha_L \\ V_U &= \frac{M}{h} \tan \alpha_U \\ V_L &= \frac{M}{h} \tan \alpha_L \\ V_w &= S - \frac{M}{h} (\tan \alpha_U + \tan \alpha_L) \end{aligned} \right\} \quad \dots \text{Eqs. 5.17}$$

We note that part of the shear, S , is now carried by the flanges due to their slopes, and due account of this should be taken when assessing the shear force in the web. Reference will be made to this later when we discuss Wagner beams.

Shear Webs

5.9. General

The transmission of shear from one part of a structure to another may be accomplished in several ways. One method would be to use a truss with diagonal members, as in Fig. 5.17 (a). An alternative method would be to replace the diagonal members by a sheet web, as in Fig. 5.17 (b). In this latter type of construction,

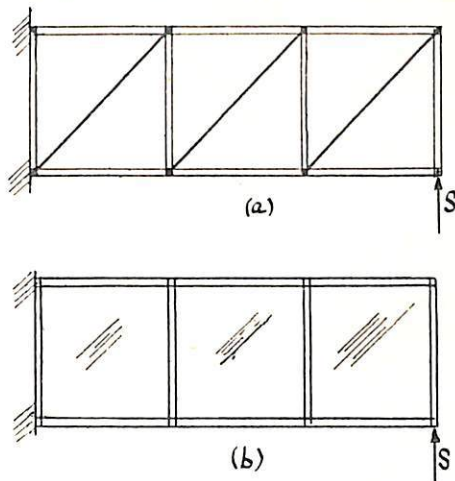


FIG. 5.17.—TRANSMISSION OF SHEAR.

which is commonly used in aircraft, the web is assumed to carry the whole of the shear whilst the flanges, or edge members, carry the bending moment as axial load, as indicated in para. 5.7. We shall now consider the webs of such beams in a little more detail, extending our discussion to cover cases where the web is not a plane web.

5.10. Stress Conditions in a Shear Web

As stated in para. 5.7, the shear flow in the web is assumed constant over a small length of the beam, and Fig. 5.18 shows this condition. In para. 3.3 (b) it is shown that pure shear gives rise to tensile and compressive stresses in two

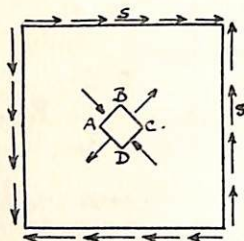


FIG. 5.18.—STRESS CONDITIONS IN PURE SHEAR.

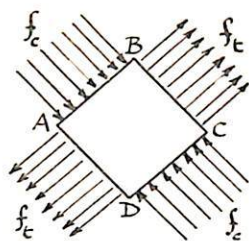


FIG. 5.19.—TENSION AND COMPRESSION ON ELEMENT OF SHEAR WEB.

perpendicular directions, at 45° to the direction of shear. The magnitude of these stresses is the same as that of the shear stress. Consequently, an element ABCD is subjected to stresses f_c and f_t as indicated in Fig. 5.19. A thin web is unable to withstand much in the way of compression before buckling, and it may be that the web will buckle into waves as indicated in Fig. 5.20. In such

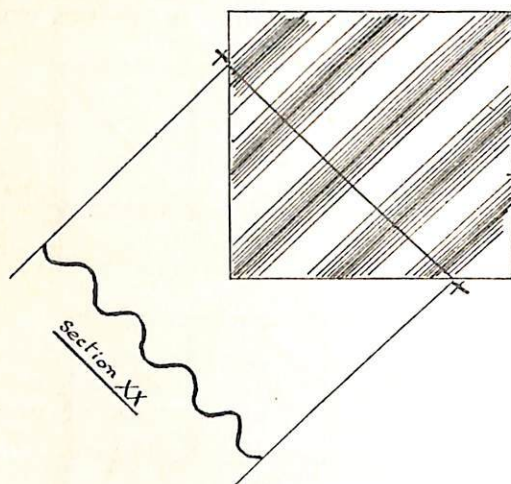


FIG. 5.20.—BUCKLING OF SHEAR WEB DUE TO DIAGONAL COMPRESSION.

a case, the shear is transmitted chiefly by the diagonal tension in the web, which acts in a manner similar to a truss with a number of diagonal members. In fact, in developing a simple theory for the transmission of shear by a web which has buckled, it is assumed that the web is incapable of carrying any compression at all, so that f_c is taken to be zero. When this state arises the web is spoken of as

a complete diagonal tension-field web. The theory of beams with webs of this type is based on the work of H. Wagner, and such beams are often called "Wagner beams." In this instance we shall concern ourselves only with webs which are of such dimensions as to withstand the compressive stress f_c without buckling, leaving the discussion of Wagner beams to a later chapter. Webs which carry the shear without buckling are termed "shear resistant webs."

5.11. Plane Webs

The case of a thin-web beam with flanges to carry the bending end loads has been discussed in para. 5.7. It will be remembered that, in the notation of Fig. 5.21, the shear flow in the web is $s = S/h$, and that at a distance L along the beam from the point of application of S the end load in a flange, P , is given by

$$P = \frac{M}{h} = \frac{S \cdot L}{h} = s \cdot L$$

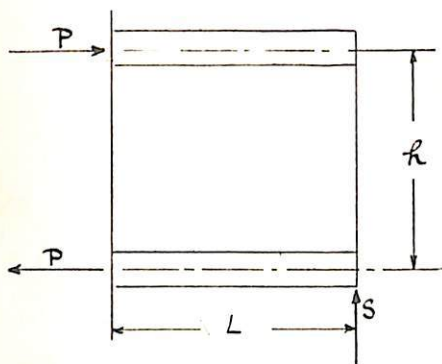


FIG. 5.21.—PLANE WEB—NOTATION.

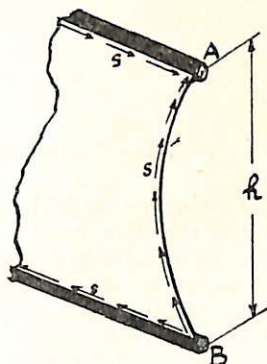


FIG. 5.22.—SHEAR FLOW IN CURVED WEB.

5.12. Curved Webs

(a) MAGNITUDE AND DIRECTION OF RESULTANT SHEAR FORCE

A shear web may be curved as in Fig. 5.22, typical examples being found in stressed-skin fuselages and wings. The shear flow will still be constant, as indicated, but we must consider how to determine the resultant shear force in magnitude, direction and location. Let any element of the web, of length δp , make an angle ϕ with an axis XX (Fig. 5.23). The shear force on this element is $s \cdot \delta p$, and the components of this force perpendicular and parallel to XX are respectively $(s \cdot \delta p) \sin \phi$ and $(s \cdot \delta p) \cos \phi$, as shown by Fig. 5.23 (b). All such elements of the web may be treated in the same way and by summation we shall obtain the following relationships.

$$\begin{aligned} \text{Component of web shear perpendicular to XX} &= \Sigma (s \cdot \delta p) \sin \phi \\ &= s \Sigma \delta p \sin \phi \\ &= s \times (\text{projected length of web perpendicular to XX}) \end{aligned}$$

Similarly the component parallel to XX = $s \times (\text{projected length of web parallel to XX})$

What is meant by "projected length of web" is illustrated in Fig. 5.24. We should refer really to the projected length of the straight line joining the ends

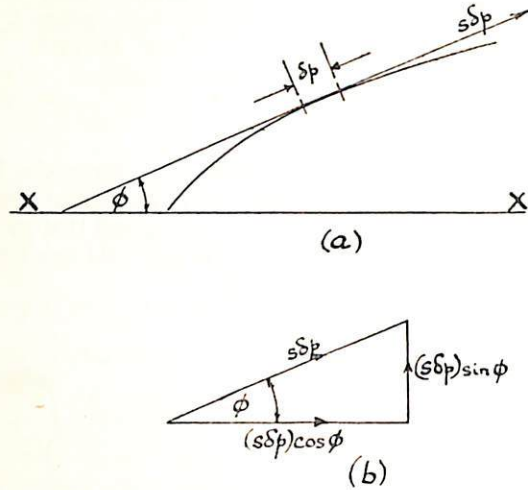


FIG. 5.23.—COMPONENTS OF SHEAR FORCE ON ELEMENT OF CURVED WEB.

of the web, so that if the component of the shear force in the direction XX is required, the ends A and B of the web are projected on to XX at A' and B' . The desired component is then equal to the shear flow times the length $A'B' = s \cdot A'B'$. This applies for the component in *any* direction. In particular, in Fig. 5.22, the component in a direction parallel to the line AB will be $s \cdot h$. Further, the dis-

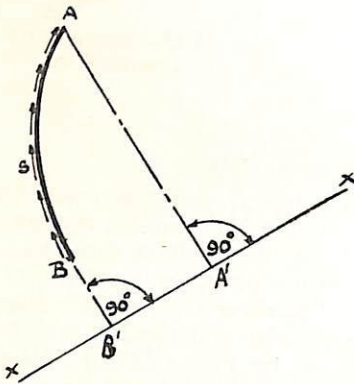


FIG. 5.24.—PROJECTED LENGTH OF CURVED WEB.



FIG. 5.25.—CURVED WEB REGARDED AS MADE UP OF STRAIGHT-LINE ELEMENTS AND RESULTANT SHEAR FORCE FROM FORCE POLYGON.

tance between the projections of A and B on to a line at right-angles to AB will be zero, so that there is no component of the web shear force perpendicular to the line joining the ends of the web. Consequently the component parallel to AB will be the resultant shear force, S , so that we may say

$$S = s \cdot h \quad \text{Eq. 5.18}$$

This may be seen quite simply in another way. Let us imagine the web to consist of a large number of straight-line elements, as in Fig. 5.25. Then, since the shear flow is constant, the shear force in each element will be proportional to the length of the element, and the drawing of the web will represent the force polygon for the web. In order to obtain a system of forces in equilibrium, a force proportional to the length AB and parallel to AB must be introduced so that the force polygon may be closed. The resultant of the forces in the web is equal and opposite to this force, so that the resultant shear in the web is proportional to AB, and is equal to $s \cdot AB$ since s is the constant of proportionality. The resultant is parallel to AB in the direction from B to A.

(b) MOMENT OF SHEAR FORCE IN A CURVED WEB ABOUT ANY POINT

So far we have determined the magnitude of the resultant shear force in a curved web as the product of the shear flow and the distance measured in a straight

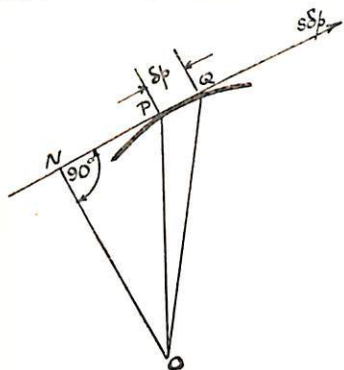


FIG. 5.26.—MOMENT OF SHEAR FORCE IN ELEMENT OF WEB ABOUT ANY POINT, O.

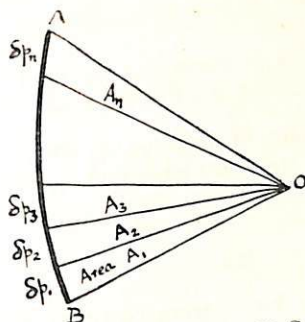


FIG. 5.27.—TOTAL MOMENT OF SHEAR IN WEB AB ABOUT O.

line between the ends of the web, the direction of the resultant being parallel to this straight line. We have not, however, determined the line of action of the resultant. To do this we must derive an expression for the moment about any point of the shear in a curved web.

We again consider an element, PQ, of length δp , so that the shear force on the element is $s \cdot \delta p$. If, in Fig. 5.26, ON is perpendicular to PQ, the moment of the shear force on PQ about O is equal to

$$M = (s \cdot \delta p) ON$$

But

$$\delta p \cdot ON = 2 \times \text{area of triangle OPQ}$$

If we proceed in this manner for all elements of the web, we conclude that the total moment of the shear force in the web about O is M_o , given by (referring to Fig. 5.27)

$$\begin{aligned} M_o &= s(2A_1 + 2A_2 + 2A_3 + \dots) \\ &= s \times 2 \times \text{area OAB} \\ &= 2A_o s \end{aligned}$$

Eq. 5.19

(c) LOCATION OF RESULTANT SHEAR FORCE

From the preceding discussion, it is obvious that the shear force in the curved web of Fig. 5.28 has a definite moment about a point O in the line AB equal

to $2A_0s$, where A_0 is the shaded area in the figure. Consequently, the resultant shear force cannot act along AB, but its line of action must be at some distance, e , to the left of AB as shown. This distance e may be determined from the fact that the moment of the resultant of a system of forces about any point is equal

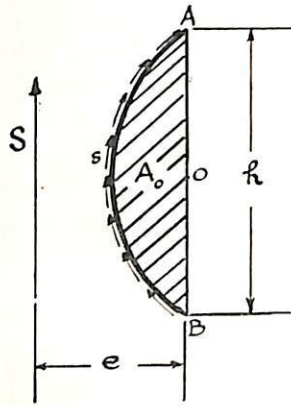


FIG. 5.28.—LOCATION OF RESULTANT SHEAR FORCE.

to the sum of the moments of all the forces of the system about that point. Hence, taking moments about O, we obtain

$$S.e = 2A_0s$$

i.e.,

$$e = \frac{2A_0s}{S}$$

But we know from equation 5.18 that $S = s.h$

Therefore,

$$e = \frac{2A_0}{h} \quad \dots \dots \dots \text{Eq. 5.20}$$

The distance e locates the "shear centre" of the web. Since the resultant shear force passes through the shear centre, it is obvious that the shear in the web has no moment about this point. Consequently to avoid any twisting due to an unbalanced couple, which would otherwise exist, any load should be applied so that it acts through the shear centre. The importance of this will be emphasised in later chapters.

CHAPTER VI

TORSION

6.1. Introduction

IN Volume I we discussed the torsion of circular sections. Whilst this is a problem of frequent occurrence in aircraft structures, as, for example, in the case of torque tubes in control circuits, there are many cases in which members of non-circular section are subjected to twisting couples. Torsion of a solid non-circular section is a complex problem, and the theoretical analysis of such members is beyond the scope of this book, but the problem will be discussed in general terms. Thin-walled sections, such as wings and fuselages, must also be considered, and we shall investigate some of the fundamental principles of the analysis of such sections subjected to torsion.

6.2. Solid Non-Circular Sections

For a circular section in torsion the direction of the shear stress at any point is perpendicular to the radius. This is true, in particular, for a point on the

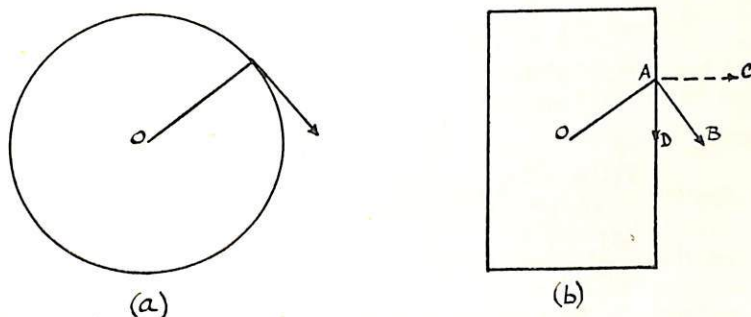


FIG. 6.1.—DIRECTION OF SHEAR AT SURFACE.

surface, as illustrated in Fig. 6.1 (a), but for any other section, such as a rectangle, this is no longer the case, as will be obvious if we consider the implications of such an assumption by reference to Fig. 6.1 (b). If the shear stress at a point A, on the surface, were perpendicular to OA, as indicated by the line AB, there would exist a component force normal to the surface in the direction AC. This cannot be the case, and we must, therefore, impose the condition that at a point on the boundary of the section the shear stress is tangential to the boundary, in the direction AD. In the case of certain simple cross-sections, such as the ellipse and the square, it is possible to solve the problem exactly by analytical methods, although the solution is too involved for inclusion here. A method of solution which is available for any cross-section is that which makes use of the *membrane analogy*, suggested by L. Prandtl. This makes use of the identity, under certain conditions, between the torsion problem and the equilibrium of a membrane of the same shape as the cross-section considered, loaded with a uniform normal pressure and stretched at the edges with a uniform tension. The importance of this method lies in the fact that it can be used experimentally. The usual

practical solution involves applying a uniform pressure to a soap film which is stretched over a hole of the required shape, cut in a thin plate, and taking the necessary measurements by optical means. The application of this analogy is as follows:—The contour lines of the deflected membrane give the direction of the shear stress at corresponding points on the cross-section of the member which is being twisted; the magnitude of the shear stress at any point is proportional to the maximum slope of the membrane at the corresponding point, this slope being measured from the plane of the edges; the volume between the plane of the edges and the deflected membrane is proportional to the applied torque.

Formulae for the maximum shear stress and the angle of twist for elliptical, square and rectangular sections are given below.

Ellipse. Major axis = $2a$; minor axis = $2b$. Applied torque = T .

$$q_{\max.} = \frac{2T}{\pi ab^2}, \quad \text{at ends of minor axis}$$

$$\theta = \frac{TL(a^2 + b^2)}{\pi G \cdot a^3 b^3}$$

where L is the overall length.

Square. Side = $2a$

$$q_{\max.} = \frac{3}{5} \cdot \frac{T}{a^3} \text{ (approx.)}, \quad \text{at mid-point of side}$$

$$\theta = \frac{4TL}{9Ga^4} \text{ (approx.)}$$

Rectangle. Long side = $2a$; short side = $2b$

$$q_{\max.} = \frac{T(15a + 9b)}{40a^2b^2} \text{ (approx.)}, \quad \text{at mid-point of long side}$$

$$\theta = \frac{40I_p \cdot TL}{A^4 \cdot G} \text{ (approx.)},$$

where I_p = polar second moment of area, and A = cross-sectional area. Inserting the values for I_p and A in the above expression, the value of the angle of twist for the rectangle may be written as

$$\theta = \frac{5(a^2 + b^2)TL}{24a^3b^3G}$$

Thin Rectangle. Approximate formulae for a thin rectangle may be derived from the above expressions. If the depth of the rectangle is $d(=2a)$, and the thickness is $t(=2b)$, such that t is small compared with d , we deduce that the maximum shear stress and the angle of twist are given approximately by,

$$\left. \begin{aligned} q_{\max.} &= \frac{3T}{dt^2} \\ \theta &= \frac{3TL}{Gdt^3} \end{aligned} \right\} \dots \dots \dots \text{Eqs. 6.1}$$

The factor, 3, in these expressions actually varies with the ratio d/t , but when d/t is large this factor tends to 3 as a limit, and this value is generally used for thin sections.

Open Sections. Sections of the type shown in Fig. 6.2 may be analysed by

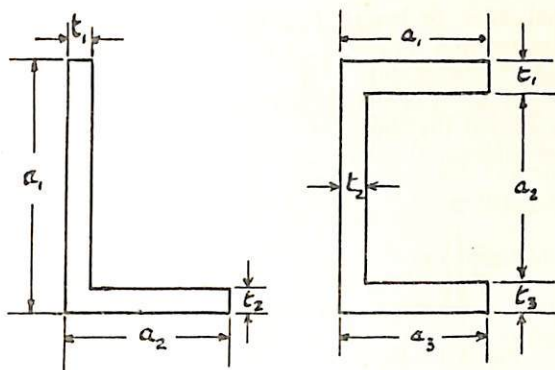


FIG. 6.2.—OPEN SECTIONS COMPOSED OF RECTANGULAR ELEMENTS.

regarding them as built up of a number of rectangular elements and applying the expressions given above. Thus, for the angle section,

$$q_{\max.} = \frac{3T}{a_1 t_1^2 + a_2 t_2^2}$$

and

$$\theta = \frac{3T}{G(a_1 t_1^3 + a_2 t_2^3)}$$

This method may be applied to any section which can be regarded as built up in this manner, but it is, of course, only approximate and it neglects entirely any stiffening effect which may be due to fillets at the corners.

6.3. Thin-walled Sections—Batho's Formula

Wings and fuselages of modern stressed-skin aircraft structures take the form of thin-walled tubes of varying cross-section. By a thin-walled tube we mean a tube in which the wall thickness is very small compared with a diametral measurement. The formulæ which will be developed in this chapter are derived from consideration of a constant section tube, but they are often applied, with a fair degree of approximation, to tubes of varying cross-section, by assuming the cross-section to remain constant over elemental lengths.

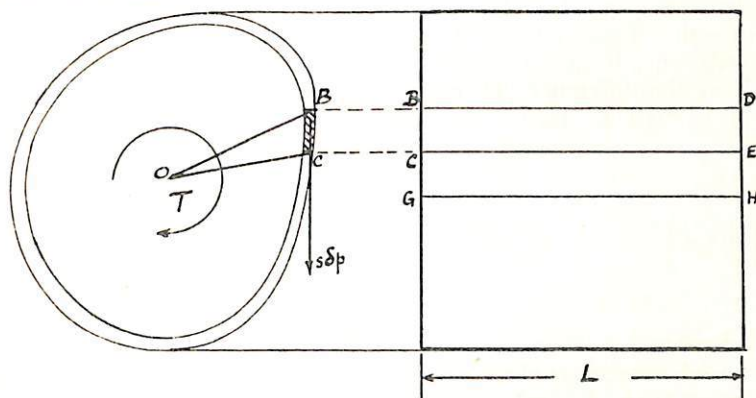


FIG. 6.3.—TORSION OF A THIN-WALLED TUBE.

If a thin-walled tube, of length L , and of constant cross-section as shown in Fig. 6.3, is subjected to a torque, T , it is obvious that there will be a tendency for adjacent cross-sections to slide over one another, giving rise to a shear flow round each cross-section. Let the shear flow thus developed on an element BC, of length δp , be s , so that the shear force on this element is $s \cdot \delta p$. Now consider the equilibrium of a longitudinal element BCED. Fig. 6.4 shows that a force

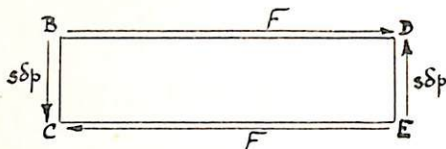


FIG. 6.4.—ELEMENT OF THIN-WALLED TUBE.

$s \cdot \delta p$ must be applied along the face ED, and forces F must be applied along the faces BD and EC for equilibrium.

We have,

$$F \cdot \delta p = (s \cdot \delta p)L$$

i.e.,

$$s = \frac{F}{L} \quad \dots \dots \dots \text{Eq. 6.2}$$

If we continue to the adjacent longitudinal element CEHG, obviously a force F is applied along CE to this new element, which in turn will apply a force of the same magnitude along GH to the next element. It follows that the force F must be the same for all longitudinal elements, there being no externally applied longitudinal force. Consequently, in equation 6.2, the expression on the right-hand side, F/L , is a constant for all positions of the element BC. Hence the shear flow, which is equal to F/L , must be constant all round the cross-section.

i.e.,

$$s = \text{constant round the section} \quad \dots \dots \dots \text{Eq. 6.3}$$

In order to determine the value of s we must equate the sum of the moments of the shear forces on all elements such as BC about some point in the plane of the cross-section to the applied torque, T . From equation 5.19, we know that the moment about any point O of the shear force on BC is the product of the shear flow and twice the area of the triangle OBC. Summing for all elements we shall obtain $T = s \cdot 2A$, where A is the total area enclosed by the median line of the thin wall. Usually it will be possible to take A as the area enclosed by the outer wall, since if the wall thickness is small compared with the overall width of the section, the difference between this area and that enclosed by the median line will be negligible. Hence, we may write

$$s = \frac{T}{2A} \quad \dots \dots \dots \text{Eq. 6.4}$$

If t is the thickness of the wall at any point, the shear stress, q , at that point is given by

$$q = \frac{s}{t} = \frac{T}{2At} \quad \dots \dots \dots \text{Eq. 6.5}$$

Equation 6.5 constitutes the *Batho* shear formula for thin-walled sections. It should be noted that it is nowhere implied that the wall thickness shall be constant round the section, the only proviso being that the wall thickness shall be small.

6.4. Angle of Twist of Thin-walled Sections

In view of the stiffness requirements for aeroplane structures, it is necessary to be able to determine the angle through which a thin-walled tube will twist under the application of a given torque. Initially we consider the constant-section torsion tube shown in Fig. 6.3, which is subjected to a torque, T , applied at one end and resisted at the other end. It is assumed that cross-sections are free to warp out of their planes, the reason for this assumption being left for discussion later.

Suppose that one end of the tube rotates through an angle θ relative to the other end. Then the element BDEC will be subjected to shear strain and will absorb shear strain energy. From equation 2.2 we know that the shear strain energy of a block whose cross-sectional area is A and whose length is L is given by

$$U_s = \frac{q^2 AL}{2G}$$

where q is the shear stress and G is the Modulus of Rigidity.

In the case under consideration, for the element BDEC, the shear stress is s/t so that we have, denoting by δU the strain energy of an element,

$$\delta U = \frac{s^2(t \cdot \delta p)L}{t^2 \cdot 2G} = s^2 \cdot \frac{\delta p}{t} \cdot \frac{L}{2G}$$

Consequently, for the whole section,

$$U = \frac{L}{2G} \oint s^2 \frac{dp}{t} \quad \dots \dots \dots \text{Eq. 6.6}$$

where the symbol $\oint$ indicates integration round the section. This strain energy must be equal to the external work done by the torque, T , which is $\frac{1}{2}T\theta$.

Hence,
$$\frac{1}{2}T\theta = \frac{L}{2G} \oint s^2 \frac{dp}{t}$$

or
$$\theta = \frac{L}{G} \oint \frac{s^2 dp}{T t} = \frac{L}{G} \oint \frac{s \cdot dp}{2A \cdot t}$$

since $T = 2As$

i.e.,
$$\theta = \frac{L}{2AG} \oint \frac{s dp}{t} \quad \dots \dots \dots \text{Eq. 6.7}$$

Alternatively, in terms of T ,

$$\theta = \frac{TL}{G \cdot 4A^2} \oint \frac{dp}{t} = \frac{TL}{G \mathcal{J}} \quad \dots \dots \dots \text{Eq. 6.8}$$

where
$$\mathcal{J} = 4A^2 \oint \frac{dp}{t}$$

The angle of twist has been expressed in two forms in equations 6.7 and 6.8. It will be found later that equation 6.7 is the more convenient form in most cases, but the expression of immediate concern is that given by equation 6.8. It will be noted that this is similar to the expression developed in para. I.12.2 for a

circular section shaft. The difference lies in the value of $\mathcal{J}$, the *torsion constant*. It may be verified quite easily that for a *thin-walled* circular cylinder the two values of $\mathcal{J}$ are the same ($2\pi r^3 t$).

Example 6.1. Determine the shear stress in each web of the section shown

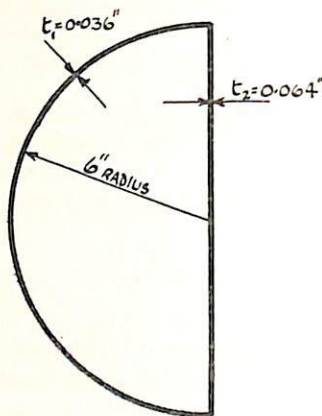


FIG. 6.5.—EXAMPLE 6.1.

in Fig. 6.5, for an applied torque of 10,000 lb. in. Calculate the angle of twist over a length of 10 ft., taking $G = 4 \times 10^6$ lb. per sq. in.

$$\text{Enclosed area} = A = \frac{\pi \cdot 6^2}{2} = 18\pi \text{ sq. in.}$$

$$\text{For curved web,} \quad \int \frac{dp}{t} = \frac{6\pi}{0.036} = 523$$

$$\text{For plane web,} \quad \int \frac{dp}{t} = \frac{12}{0.064} = 187.5$$

$$\therefore \oint \frac{dp}{t} = 523 + 187.5 = 710.5$$

$$\text{Shear flow} = s = \frac{T}{2A} = \frac{10,000}{36\pi} = 88.3 \text{ lb. per in.}$$

$$\text{Shear stress in curved web} = \frac{s}{t_1} = \frac{88.3}{0.036} = 2453 \text{ lb. per sq. in.}$$

$$\text{,, ,, ,, plane ,,} = \frac{s}{t_2} = \frac{88.3}{0.064} = 1380 \text{ lb. per sq. in.}$$

$$\mathcal{J} = \frac{4A^2}{\oint \frac{dp}{t}} = \frac{4 \times (18\pi)^2}{710.5} = 18.02 \text{ in.}^4$$

$$\theta = \frac{TL}{G\mathcal{J}} = \frac{10,000 \times 120}{4 \times 10^6 \times 18.02} = 0.01664 \text{ radian} = 0.955^\circ$$

(Note that L must be expressed in inches, in order to keep the units consistent.)

6.5. Torsion of Multi-cell Thin-walled Tubes

In stressed-skin aircraft structures the multi-cell torsion tube is frequently encountered. A typical example is a two-spar stressed-skin wing, in which the entire section from the leading edge to the rear spar is effective in carrying torsion (Fig. 6.6). Several different methods of analysing such a tube subjected to torsion

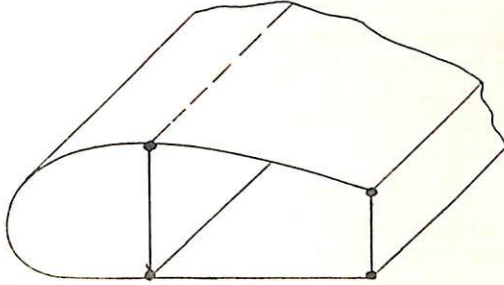


FIG. 6.6.—TWO-CELL TORSION TUBE.

have been suggested, and two such methods will be presented here. The first is the more fundamental method, but the second may be found more convenient for the purposes of calculation. In both cases we assume the section to be "idealised," in that the flange material is assumed to be concentrated, as indicated, and we assume further that the tube is of constant cross-section and free from any restraint against warping of the section. For convenience we shall adopt a form of Bow's notation as shown in Fig. 6.7, so that we can use suffices to indicate

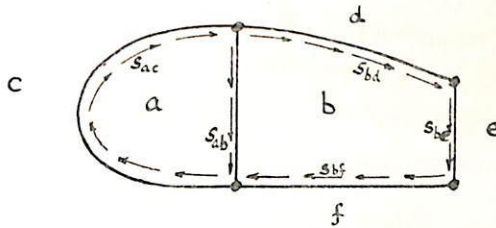


FIG. 6.7.—NOTATION FOR TWO-CELL TORSION TUBE.

which particular portion of the wall we are considering—for example, t_{ac} denotes the thickness of the nose skin, p_{ac} denotes the perimeter of the nose skin, t_{ab} denotes the thickness of the front spar web, p_{ab} denotes the depth of the front spar web, and so on. Similarly, A_a is the enclosed area of the cell a , and A_b is that of cell b .

In each portion of the skin the shear flow will be constant—this can be proved as in para. 6.3. Since there is no bending (we are considering the tube loaded in torsion only), there will be no end load in any of the flanges, so that the algebraic sum of the shear flows in the webs meeting at any flange must be zero, in view of the results obtained in para. 5.7. With the directions of shear flow as given in the figure, this condition yields the following equations:—

$$s_{ac} - s_{ab} - s_{bd} = 0$$

$$s_{bd} - s_{be} = 0$$

$$s_{be} - s_{bf} = 0$$

$$s_{bf} + s_{ab} - s_{ac} = 0$$

From these four equations we obtain,

$$s_{bd} = s_{be} = s_{bf} = s_b \text{ (say)}$$

and

$$s_{ab} = s_{ac} - s_b \quad \dots \quad \text{Eq. 6.9}$$

i.e., we obtain finally one equation connecting three unknown quantities. A second equation may be obtained from the fact that the internal shear flows must balance the applied torque, T . In the figure, the shear flows are represented as reactions to the applied torque, and taking moments about any point in the front spar web, for convenience, we have,

$$2A_a s_{ac} + 2A_b s_b = T \quad \dots \quad \text{Eq. 6.10}$$

It will be noted that taking moments about any other point would give an equation which would reduce to equation 6.10 on application of the relationships already obtained between s_{ac} , s_{ab} and s_b .

In order to derive the third equation necessary for complete solution we apply the principle of minimum strain energy, and to do this we must first determine the strain energy in shear of the shell. For unit length of a shear web, of depth p and thickness t , subjected to a shear flow of magnitude s , the strain energy in shear may be written as $\frac{s^2}{2G} \cdot \frac{p}{t}$. Assuming, for simplicity, that the value of G is the same for all webs, the total strain energy per unit length, U , is

$$U = \frac{1}{2G} \left[(s_{ac})^2 \left(\frac{p}{t} \right)_{ac} + (s_{ab})^2 \left(\frac{p}{t} \right)_{ab} + s_b^2 \left\{ \left(\frac{p}{t} \right)_{bd} + \left(\frac{p}{t} \right)_{be} + \left(\frac{p}{t} \right)_{bf} \right\} \right]$$

Treating s_{ac} as the redundancy, by the principle of minimum strain energy we must have $\frac{\partial U}{\partial s_{ac}} = 0$. Since there is only one *independent* variable, we may use the total differential, d , instead of the partial, ∂ .

Performing the differentiation, and writing $\alpha = \frac{p}{t}$, we have

$$\frac{dU}{ds_{ac}} = \frac{1}{2G} \left[2s_{ac}\alpha_{ac} + 2s_{ab}\alpha_{ab} \frac{ds_{ab}}{ds_{ac}} + 2s_b(\alpha_{bd} + \alpha_{be} + \alpha_{bf}) \frac{ds_b}{ds_{ac}} \right] = 0 \quad \dots \quad \text{Eq. 6.11}$$

Now

$$s_b = \frac{T - 2A_a s_{ac}}{2A_b}$$

and

$$s_{ab} = s_{ac} - s_b = s_{ac} \left(1 + \frac{A_a}{A_b} \right) - \frac{T}{2A_b}$$

so that

$$\frac{ds_b}{ds_{ac}} = -\frac{A_a}{A_b}$$

and

$$\frac{ds_{ab}}{ds_{ac}} = 1 + \frac{A_a}{A_b}$$

Substituting these values in equation 6.11, our equation becomes,

$$s_{ac}\alpha_{ac} + s_{ab}\alpha_{ab} \left(1 + \frac{A_a}{A_b} \right) - s_b(\alpha_{bd} + \alpha_{be} + \alpha_{bf}) \frac{A_a}{A_b} = 0$$

This reduces to

$$\frac{s_a \alpha_{ac} + s_{ab} \alpha_{ab}}{A_a} - \frac{s_b (\alpha_{bd} + \alpha_{be} + \alpha_{bf}) - s_{ab} \alpha_{ab}}{A_b} = 0 \quad \text{Eq. 6.12}$$

Equations 6.9, 6.10 and 6.12 can now be solved to give the shear flows in terms of the applied torque, T .

Alternative Method. In this method we assume that the shell can be split into two distinct torsion tubes, the common wall, ab , being included in both cells with its true shear flow. If we had a single cell, a , subjected to torsion, there would be a constant shear flow, s_a , round it; similarly torsion applied to cell b alone would produce a constant shear flow, s_b , around that cell. Assuming the same effect to be produced when torsion is applied to the two-cell shell, the resultant shear flow in the common wall, ab , will be $(s_a - s_b)$ when the wall is regarded as part of the cell a , and $(s_b - s_a)$ in the opposite direction when the wall is taken as part of cell b (Fig. 6.8). This assumption of a constant shear flow round each cell, with a resultant shear flow in the common wall equal to the difference between those two flows, agrees with equation 6.9 obtained in the first analysis. This means that in making this assumption we have automatically used the relationships leading to equation 6.9, and we have only two unknowns, so that we need two equations for a solution. These could be obtained just as equations 6.10 and 6.12 were obtained, but, of these, we shall use only equation 6.10 in the original form, and inserting the quantities now considered, we write

$$2A_a s_a + 2A_b s_b = T \quad \text{Eq. 6.13}$$

The remaining equation required is derived by observing that, in order to preserve continuity of the section, each cell must twist through the same angle, which gives

$$\theta_a = \theta_b \quad \text{Eq. 6.14}$$

All that remains now is to express equation 6.14 in terms of s_a and s_b . From equation 6.7,

$$\theta = \frac{L}{2AG} \oint s \cdot \frac{ds}{t}$$

Considering unit length, so that $L = 1$, we write,

$$\theta_a = \frac{1}{2A_a G} [s_a \alpha_{ac} + (s_a - s_b) \alpha_{ab}] \quad \text{Eq. 6.15}$$

and

$$\theta_b = \frac{1}{2A_b G} [s_b (\alpha_{bd} + \alpha_{be} + \alpha_{bf}) + (s_b - s_a) \alpha_{ab}] \quad \text{Eq. 6.16}$$

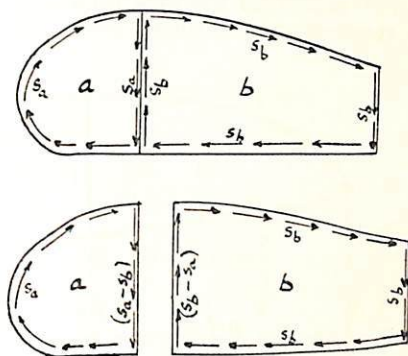


FIG. 6.8.—TWO-CELL TUBE SPLIT INTO TWO COMPONENT TUBES.

Substituting these values in equation 6.14,

$$\frac{1}{2A_a G}[s_a \alpha_{ac} + (s_a - s_b) \alpha_{ab}] = \frac{1}{2A_b G}[s_b(\alpha_{ba} + \alpha_{be} + \alpha_{bf}) + (s_b - s_a) \alpha_{ab}] \quad \text{Eq. 6.17}$$

Solution of equations 6.13 and 6.17 yields s_a and s_b in terms of T . It will be observed that equations 6.12 and 6.17 are actually the same, if use is made of equation 6.9.

The second method has the advantage that it contains the expressions required for determination of the angle of twist of the section if that should be desired.

A numerical example follows to illustrate the application of this method to a two-cell torsion tube.

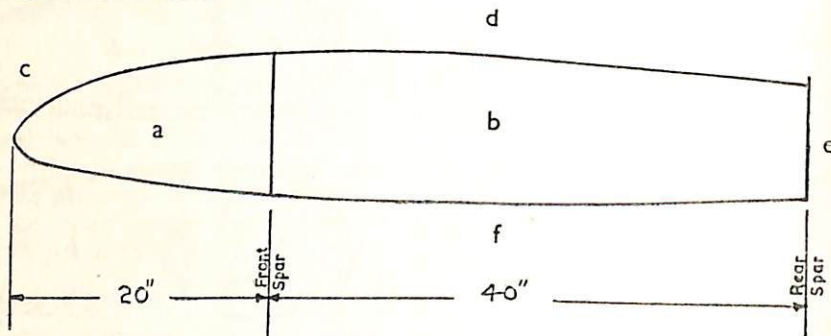


FIG. 6.9.—EXAMPLE 6.2.

Example 6.2. For the wing section shown in Fig. 6.9, the following data are given (notation as in para. 6.5) :—

$$A_a = 166 \text{ sq. in.} \quad A_b = 435 \text{ sq. in.}$$

$$p_{ac} = 46 \text{ in.} \quad p_{ab} = 11.5 \text{ in.} \quad p_{bd} = p_{bf} = 42 \text{ in.} \quad p_{be} = 9.15 \text{ in.}$$

$$t_{ac} = 0.036 \text{ in.} \quad t_{ab} = 0.064 \text{ in.} \quad t_{bd} = t_{bf} = 0.036 \text{ in.} \quad t_{be} = 0.048 \text{ in.}$$

Determine the shear stresses and the angle of twist per unit length for an applied torque of 200,000 lb. in., taking $G = 4 \times 10^6$ lb. per sq. in. Using the notation introduced earlier, we calculate

$$\alpha_{ac} = \frac{46}{0.036} = 1277$$

$$\alpha_{ab} = \frac{11.5}{0.064} = 179.5$$

$$\alpha_{bd} = \alpha_{bf} = \frac{42}{0.036} = 1167$$

$$\alpha_{be} = \frac{9.15}{0.048} = 190.6$$

Now equation 6.13 becomes in this case,

$$332s_a + 870s_b = 200,000 \quad . \quad . \quad . \quad . \quad (1)$$

From equation 6.15

$$\theta_a = \frac{1}{332G}[1277s_a + 179.5(s_a - s_b)]$$

$$= \frac{1}{G}(4.385s_a - 0.541s_b) \quad \dots \quad (2)$$

From equation 6.16

$$\begin{aligned} \theta_b &= \frac{1}{870G}[(1167 + 190.6 + 1167)s_b + 179.5(s_b - s_a)] \\ &= \frac{1}{G}(3.108s_b - 0.206s_a) \quad \dots \quad (3) \end{aligned}$$

Then equation 6.17 gives,

$$\begin{aligned} 4.385s_a - 0.541s_b &= 3.108s_b - 0.206s_a \\ \text{which reduces to} \quad s_a &= 0.795s_b \quad \dots \quad (4) \end{aligned}$$

From (1) and (4) we now have

$$264s_b + 870s_b = 200,000$$

$$\text{i.e.,} \quad s_b = 176.4 \text{ lb. per in.}$$

$$\text{then it follows that} \quad s_a = 140.2 \text{ lb. per in.}$$

The shear stresses may be calculated from the relationship, $q = \frac{s}{t}$

Thus,

$$q_{ac} = \frac{s_a}{t_{ac}} = \frac{140.2}{0.036} = 3890 \text{ lb. per sq. in.}$$

$$q_{ab} = \frac{s_b - s_a}{t_{ab}} = \frac{36}{0.064} = 563 \text{ lb. per sq. in.}$$

$$q_{bd} = q_{bf} = \frac{s_b}{t_{bd}} = \frac{176.4}{0.036} = 4890 \text{ lb. per sq. in.}$$

$$q_{be} = \frac{s_b}{t_{be}} = \frac{176.4}{0.048} = 3670 \text{ lb. per sq. in.}$$

The angle of twist per unit length is determined from (2) or (3) as

$$\begin{aligned} \theta = \theta_a = \theta_b &= \frac{1}{4 \times 10^6}(4.385 \times 140.2 - 0.541 \times 176.4) \\ &= 0.1298 \times 10^{-3} \text{ radians} \\ &= 7.44 \times 10^{-3} \text{ degrees} \end{aligned}$$

6.6. Twisting of Wings. Effect of Root Restraint

Aeroplane wings of stressed skin construction are generally tapered both in chord and thickness. Moreover, the cross-sections are not necessarily geometrically similar along the span. The theories which have been developed in the preceding paragraphs all assume constant cross-section, and freedom from axial constraint. Torsional stresses in and twisting of a tapered shell, such as a wing, may be computed approximately by splitting up the shell into elementary lengths, over each of which the section may be assumed constant, and then applying the methods developed in the preceding paragraphs to each element in turn. The total angle of twist for the whole shell is then the sum of the angles of twist for the several elements.

The effects of constraint against axial movement, such as occurs at the root

of a wing, can be computed, and in general such effects do not extend very far outboard from the root. In order to indicate how constraint against axial movement can influence the stresses set up in a torsion shell, we consider the rectangular box shown in Fig. 6.10, subjected to a torque T . Initially, we assume the box

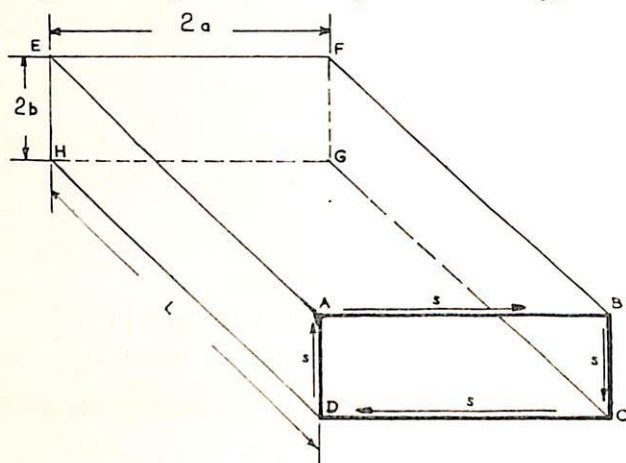


FIG. 6.10.—RECTANGULAR TORSION TUBE.

to be free to move axially at both ends. The shear flow, s , is constant round the section, and is equal to $T/2.4ab$. We may regard any distortion of the box as being due to two distinct actions—i.e., distortion due to shear of the panels, without rotation, and distortion due to rotation without shear deformation of the

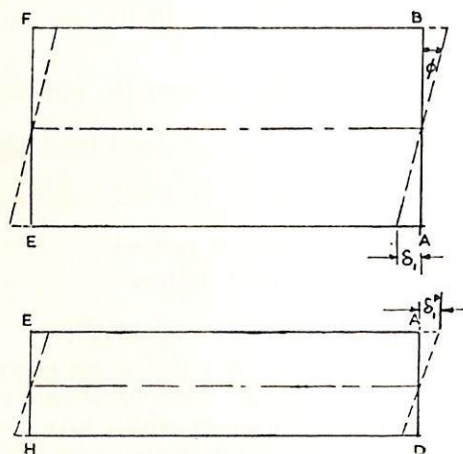


FIG. 6.11.—DISTORTION OF PANELS OF RECTANGULAR TUBE DUE TO SHEAR ALONE.

panels. Both of these actions will induce axial movement of a point such as A. If we evaluate the axial movement of A, considering it as a point on panel ABFE and then as a point on panel ADHE, we shall have the condition that the two movements thus calculated must be equal in order to preserve continuity of the structure. Fig. 6.11 shows the distortion of these two panels due to shear. The axial movement of the point A on panel ABFE, denoted by δ_1 , is, for small values

of shear strain, $-a\phi$, measuring positive movement in the direction EA. But ϕ , the shear strain, is equal to the shear stress divided by G , so that

$$\delta_1 = -\frac{as}{Gt_1}$$

Similarly, the axial movement of A on panel ADHE due to shear is δ'_1 , where

$$\delta'_1 = +\frac{bs}{Gt_2}$$

If the end EFGH is assumed held against rotation, the effect of twist alone is

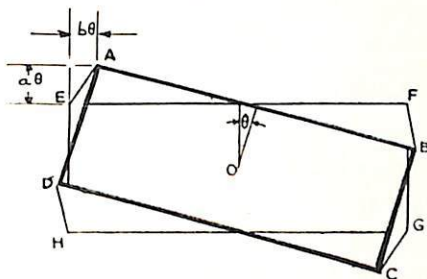


FIG. 6.12.—EFFECT OF TWISTING ALONE ON RECTANGULAR TUBE.

as shown in Fig. 6.12, and it will be observed that there is in this view lateral movement of the point A along DA and AB. Regarding now the displacement of panels ABFE and ADHE due to rotation only, assuming that the mid-points of the edges AB, FE, AD and HE do not move, we obtain Fig. 6.13. From

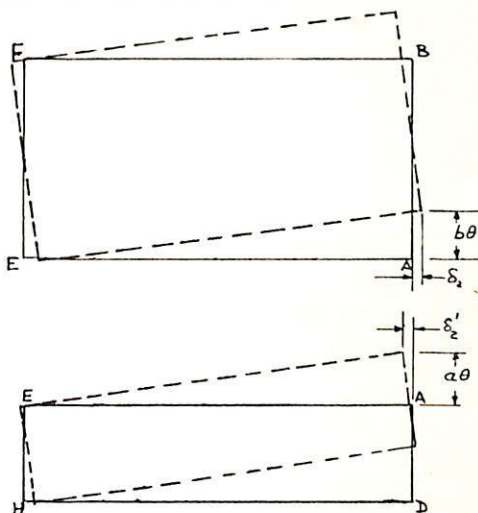


FIG. 6.13.—DISPLACEMENT OF PANELS OF RECTANGULAR TUBE DUE TO ROTATION ALONE.

this it will be seen that further axial movements of A are produced. Denoting these by δ_2 and δ'_2 , as indicated, we see that

$$\delta_2 = +\frac{ab\theta}{L} \quad \text{and} \quad \delta'_2 = -\frac{ab\theta}{L}$$

The total axial movement of A on ABFE is therefore $\left(-\frac{as}{Gt_1} + \frac{ab\theta}{L}\right)$, and on ADHE it is $\left(+\frac{bs}{Gt_2} - \frac{ab\theta}{L}\right)$. Since the point A must have the same axial movement whether it is regarded as a point on ABFE or on ADHE, we have

$$-\frac{as}{Gt_1} + \frac{ab\theta}{L} = \frac{bs}{Gt_2} - \frac{ab\theta}{L}$$

Hence,

$$\frac{\theta}{L} = \frac{s}{2abG} \left[\frac{a}{t_1} + \frac{b}{t_2} \right] \quad \text{Eq. 6.18}$$

Putting $s = \frac{T}{8ab}$, this becomes

$$\frac{\theta}{L} = \frac{T}{G} \left\{ \frac{a}{t_1} + \frac{b}{t_2} \right\} = \frac{T}{G} \frac{\left\{ 4a + \frac{4b}{t_2} \right\}}{4(16a^2b^2)} = \frac{T}{G} \oint \frac{dp}{t}$$

i.e., equation 6.18 is the same as the equation developed in para. 6.4 for the angle of twist with the assumption that the cross-section is free to warp. It will be obvious that if axial movement is prevented, the effect will be to produce bending of the panels, and, if we assume bending to be taken by flanges at the corners of the box, end loads will be produced in these flanges which would not exist if restraint were not present.

Let us determine under what conditions the cross-section will not warp. If there is to be no warping, the resultant axial movement of the point A must be zero.

i.e.,

$$\delta_1 + \delta_2 + \delta'_1 + \delta'_2 = 0$$

$$-\frac{as}{Gt_1} + \frac{bs}{Gt_2} = 0$$

or

$$\frac{a}{t_1} = \frac{b}{t_2}$$

If this relationship is satisfied, there will be no warping of the cross-section, and root constraint will have no effect. In general, unless this relationship is satisfied, it will be necessary to take account of the flange end loads induced by root constraint, but these effects are serious only in the neighbourhood of the root.

CHAPTER VII

COMBINED BENDING AND TORSION OF THIN-WALLED SECTIONS

7.1. Preliminary Assumptions

THE type of structure to be considered in this chapter is the "skin-stringer" type, which we shall idealise by assuming that the skin carries only shear, all end loads, both tensile and compressive, being taken by the stringers. In practice, the skin will be capable of carrying tension and some of the skin will be effective in carrying compression, but for an initial approach to the problem, the simplifying assumption stated above is made.

7.2. Determination of Shear Flow—Single-cell Section

Fig. 7.1 (a) represents a length L of a stiffened shell, subjected to bending, shear and torsion. The torsion is represented by the displacement of the shear force, S , from the cross-section in Fig. 7.1 (b).

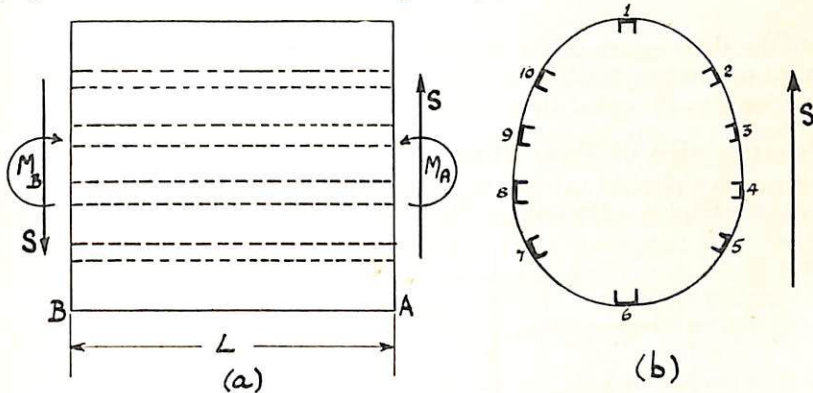


FIG. 7.1.—STIFFENED SHELL.

We know that we can obtain the shear flow in the various skin panels in terms of one unknown shear flow purely from consideration of the end load increments in the longitudinal stringers over the length L , the shear force, S , being considered constant over this length. Accordingly, we need one further equation to determine the unknown shear flow. (It is, of course, assumed that bending is taking place about a principal axis, or that due account has been taken of the effects of any unsymmetrical bending which may be present, in evaluating the end load increments in the stringers.) This required equation is obtained by consideration of the torque on the cross-section. If we take any point O in the plane of the section, and join this point to the stringers 1, 2, . . . 10, by straight lines as in Fig. 7.2, then the moment of the shear flow in panel 1 about O is equal to $2s_1A_1$, where A_1 is the area $O12$, and the moments about O of the shears in the other panels are given by similar expressions. If the point O be at distance d from the line of action of S , we must have, for equilibrium,

$$2s_1A_1 + 2s_2A_2 + 2s_3A_3 \dots + 2s_{10}A_{10} = Sd \quad \text{Eq. 7.1}$$

Since we know $s_2, s_3 \dots s_{10}$ in terms of s_1 , this equation now serves to determine s_1 .

The angle of twist follows from application of equation 6.7. If it is desired

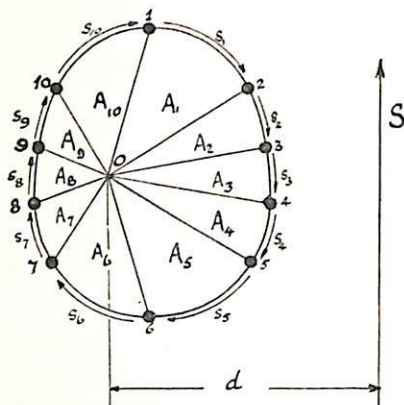


FIG. 7.2.—BALANCE OF TORQUE ON CROSS-SECTION OF STIFFENED SHELL.

to locate the shear centre of the section—i.e., the point at which a shear S must be applied to produce bending without twist—the shear flow is determined from the equation $\theta = 0$, and d then follows from equation 7.1.

7.3. Determination of Shear Flow in Two-cell Shell

This case is typical of a two-spar stressed skin wing, and the complete solution is somewhat complex. However, a simple approach to the problem is as follows.

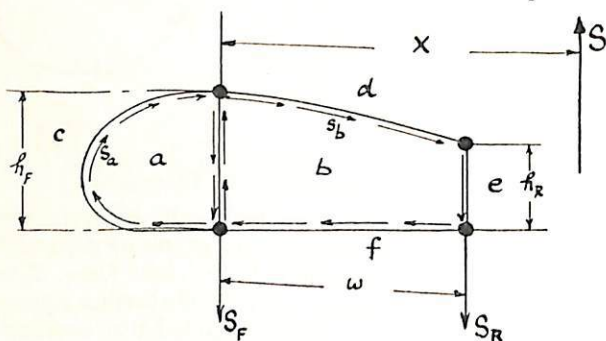


FIG. 7.3.—TWO-CELL SHELL SUBJECTED TO BENDING AND TORSION.

We assume a constant shear flow round each cell, of magnitudes s_a and s_b respectively, as for the two-cell shell in torsion, together with a shear force S_F in the front spar web and a shear force S_R in the rear spar web, as indicated in Fig. 7.3.

The resultant shear flow in the front spar web is then $\left(\frac{S_F}{h_F} + s_a - s_b\right)$ and that in the rear spar web is $\left(\frac{S_R}{h_R} + s_b\right)$. We have now the following equations:—

Taking moments about the front spar web,

$$2A_a s_a + 2A_b s_b + S_R v = SX \quad \text{Eq. 7.2}$$

Equating the angles of twist of the two cells,

$$\begin{aligned} \frac{1}{2A_a G} \left[s_a \alpha_{ac} + \left(s_a - s_b + \frac{S_F}{h_F} \right) \alpha_{ab} \right] \\ = \frac{1}{2A_b G} \left[s_b \alpha_{bd} + \left(s_b + \frac{S_R}{h_R} \right) \alpha_{be} + s_b \alpha_{bf} + \left(s_b - s_a - \frac{S_F}{h_F} \right) \alpha_{ba} \right] \quad \text{Eq. 7.3} \end{aligned}$$

Equating the vertical forces to zero,

$$S_F + S_R - S = 0 \quad \text{Eq. 7.4}$$

The remaining equation required for the solution comes from the relationship between the flange end load increments and the shear flows in the adjoining webs.

As an approximation, it is sometimes convenient to assume that the whole of the bending is taken by the spars alone and the whole torsion taken by the skin. In this case it is usual to assume that the shear, S , is carried by the two spars in proportion to their second moments of area. This means simply that we assume

$$S_F = \frac{SI_F}{I_F + I_R} \quad \text{and} \quad S_R = \frac{SI_R}{I_F + I_R} \quad \text{Eqs. 7.5}$$

This is equivalent to the assumption that the two spars deflect equally. If we locate a point F between the two spars such that it divides the distance between them inversely in the ratio of their second moments of area, a force S applied there would be equivalent to forces S_F and S_R as defined by equation 7.5. This point F may be called the *shear centre* of the two spars, and, considering the spars alone, neglecting the skin entirely, a force applied there would produce pure bending with no twist. When the skin is taken into account it is no longer true to say that a force applied at the point F , defined as above, will produce bending without twist, so that F is not a true shear centre for the whole section.

The following numerical example illustrates the approximate procedure suggested above.

Example 7.1. The wing section shown in Fig. 7.4 is the same as that in

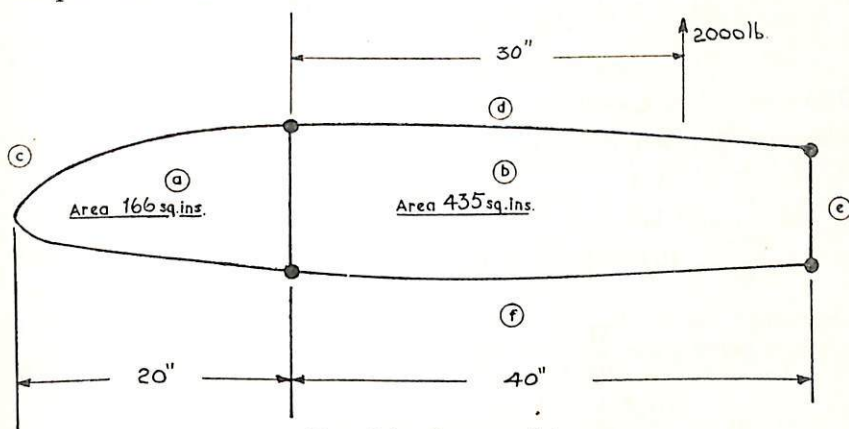


FIG. 7.4.—EXAMPLE 7.1.

Fig. 6.9, and the following additional information is given :—

Area of each flange of front spar = $A_{f_F} = 1.8$ sq. in.

Area of each flange of rear spar = $A_{f_R} = 1.6$ sq. in.

If a load of 2000 lb. is applied at 30 in. behind the front spar, determine the shear stresses in the skin panels and the spar webs.

The second moments of area of the two spars are calculated as follows.

$$I_F = \frac{1.8(11.5)^2}{2} = 119 \text{ in.}^4$$

$$I_R = \frac{1.6(9.15)^2}{2} = 67 \text{ in.}^4$$

Then,

$$S_F = 2000 \times \frac{119}{186} = 1280 \text{ lb.}$$

$$S_R = 2000 \times \frac{67}{186} = 720 \text{ lb.}$$

Equation 7.2 becomes in this case

$$332s_a + 870s_b + 720 \times 40 = 2000 \times 30$$

$$\text{i.e.,} \quad s_a + 2.62s_b = 94 \quad \dots \dots \dots (1)$$

Using the values of α for the various panels as obtained in the example in para. 6.5, equation 7.3 may be written

$$\begin{aligned} & \frac{1}{332G} \left[1277s_a + 179.5 \left(s_a - s_b + \frac{1280}{11.5} \right) \right] \\ &= \frac{1}{870G} \left[1167s_b + 190.6 \left(s_b + \frac{720}{9.15} \right) + 1167s_b + 179.5 \left(s_b - s_a - \frac{1280}{11.5} \right) \right] \end{aligned}$$

$$\text{i.e.,} \quad 4.385s_a - 0.541s_b + 60.25 = 3.108s_b - 0.206s_a + 17.23 - 23.0$$

$$4.591s_a - 3.649s_b + 66.02 = 0$$

$$s_a - 0.795s_b + 14.38 = 0 \quad \dots \dots \dots (2)$$

From (1) and (2),

$$3.415s_b = 108.38$$

$$s_b = 31.75 \text{ lb. per in.}$$

Then

$$s_a = 94 - 83.2 = 10.8 \text{ lb. per in.}$$

The shear stresses now follow as

$$q_{ac} = \frac{10.8}{0.036} = 300 \text{ lb. per sq. in.}$$

$$q_{ab} = \frac{10.8 - 31.75 + 111.3}{0.064} = 1412 \text{ lb. per sq. in.}$$

$$q_{bd} = q_{bf} = \frac{31.75}{0.036} = 882 \text{ lb. per sq. in.}$$

$$q_{be} = \frac{31.75 + 78.7}{0.048} = 2300 \text{ lb. per sq. in.}$$

LOCATION OF SHEAR CENTRE

The shear centre is found by determining the distance X (Fig. 7.3) so that the angle of twist is zero. We then have, instead of the one equation 7.3, the two equations $\theta_a = \theta_b = 0$, since a load applied at the shear centre will produce zero twist. If it is desired to evaluate the shear flows for a number of loading conditions, the most convenient way of tackling the problem may be to locate the shear centre and to determine separately the shear flows due to (i) a load S_1 at the shear centre, and (ii) a torque T_1 about the shear centre. Any other load can be represented as a load S_2 at the shear centre, together with a torque T_2 about the shear centre. Then the shear flow due to S_2 at the shear centre is that due to S_1 at the shear centre multiplied by the ratio S_1/S_2 , and the shear flow due to a torque T_2 is that due to a torque T_1 multiplied by T_1/T_2 . Since a load at the shear centre produces bending without twist, and a torque produces twist without bending, the shear flows due to these two conditions of loading may be added directly to give the resultant shear flow when the load and the torque act simultaneously.

As a numerical illustration the shear centre for the section of Fig. 7.4 will be located.

Example 7.2. The section is that of the previous example. Let the shear centre be at distance d behind the front spar. Then for a load of 2000 lb. at the shear centre, equation (1) of example 7.1 becomes,

$$s_a + 2.62s_b + 86.75 = 6.025d \quad . \quad . \quad . \quad (1)$$

Since the load is applied at the shear centre, $\theta_a = \theta_b = 0$

$$\text{i.e.,} \quad 4.385s_a - 0.541s_b + 60.25 = 0 \quad . \quad . \quad . \quad (2)$$

$$- 0.206s_a - 3.108s_b - 5.77 = 0 \quad . \quad . \quad . \quad (3)$$

Rewriting (2) and (3),

$$s_a - 0.1234s_b + 13.76 = 0 \quad . \quad . \quad . \quad (2a)$$

$$- s_a + 15.087s_b - 27.90 = 0 \quad . \quad . \quad . \quad (3a)$$

From (2a) and (3a)

$$14.9636s_b - 14.14 = 0$$

$$s_b = 0.945 \text{ lb. per in.} \quad . \quad . \quad . \quad (4)$$

$$\text{Then} \quad s_a = -13.76 + 0.117 = -13.643 \text{ lb. per in.} \quad . \quad . \quad . \quad (5)$$

Substituting these values in (1) we obtain

$$6.025d = -13.643 + 2.476 + 86.75$$

$$d = 12.55 \text{ in.}$$

Hence, a load applied at 12.55 in. behind the front spar will produce bending of the wing without twist, and the shear flows corresponding to a load of 2000 lb. at this point are given by (4) and (5) above.

The loading condition of example 7.1 is equivalent to a load of 2000 lb. at the shear centre together with a torque of 34,900 lb. in.

$$(2000 \times (30 - 12.55) = 34,900).$$

From the example in para. 6.5, a torque of 200,000 lb. in. produces shear flows $s_a = 140.2$ lb. per in., and $s_b = 176.4$ lb. per in. Hence a torque of 34,900 lb. in. produces shear flows given by

$$s_a = 140.2 \times \frac{34,900}{200,000} = 24.45 \text{ lb. per in.}$$

$$s_b = 176.4 \times \frac{34,900}{200,000} = 30.78 \text{ lb. per in.}$$

The resultant shear flows due to a load of 2000 lb. at the shear centre and a torque of 34,900 lb. in. will then be

$$s_a = -13.643 + 24.45 = 10.807 \text{ lb. per in.}$$

$$s_b = 0.945 + 30.78 = 31.725 \text{ lb. per in.}$$

These figures agree within the limits of slide rule error with those obtained in Example 7.1.

7.4. Contribution of Skin to Bending Strength

In the approximate analysis of the preceding paragraphs any contribution of the skin to strength in bending has been neglected, the whole of the bending material being assumed to be concentrated in the longitudinal flange elements. In practice, of course, this simple condition does not arise. On the tension side the skin may generally be assumed to be fully effective, and on the compression side the skin will be effective until buckling occurs. Even after the stress at which buckling occurs has been exceeded, some of the skin will continue to work with the flange elements, the amount depending on the spacing of the flanges, the buckling stress of the skin panel, and on the stress in the flange. The section should, therefore, be treated as a whole rather than be thought of as consisting of spars which carry all the bending and skin which only carries shear. The problem then becomes somewhat complex and usually requires treatment by some method of continuous approximation. The general problem is beyond the scope of this book, but it is important to realise that the approximate methods given here are insufficient for complete analysis, although they may often be found useful for preliminary rough design checks.

7.5. Effect of Skin Buckling

As explained in para. 5.10, the skin may buckle at quite low values of the shear stress, and even if we make the simplification of neglecting the skin in bending, the shear flows obtained by the methods outlined above will be affected by shear buckling of the skin. This may be taken account of—again approximately—by assuming that complete diagonal tension fields are formed in the skin panels, the effective shear modulus then becoming theoretically $\frac{5}{8}G$ (see para. 13.6); instead of using this effective value of the modulus of rigidity, it is sometimes the practice to use an equivalent thickness, t_e , equal to $\frac{5}{8}$ of the actual thickness.

BEAM DEFLECTIONS

8.1. General

IN Volume I we considered the strength of simple beams, but in general it is necessary to investigate the stiffness of a beam as well as its strength. It may often be the case that stiffness is equally as important as strength, and in this chapter we shall discuss means of determining the deflection of simple beams. The deflection due to bending only will be discussed in detail, this usually being much greater than the deflection due to shear, though in certain types of beams the latter must be taken into account.

8.2. Basic Equation for Bending

It can be shown that the radius of curvature, R , at a point (x, y) of a curve is given by

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}} \quad \text{Eq. 8.1}$$

(The quantity $1/R$ is known as the *curvature*.)

In the theory which follows it will be assumed

- (1) that deflections are small compared with the length of a beam
- (2) that slopes are small (i.e., dy/dx is small).

From the second assumption it follows that $(dy/dx)^2$ may be taken as negligible, so that equation 8.1 becomes

$$\frac{1}{R} = \frac{d^2y}{dx^2} \quad \text{Eq. 8.2}$$

We have shown already (I.11.5) that $1/R = M/EI$

Hence

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \quad \text{Eq. 8.3}$$

This is the basic equation for bending of a beam, and if M/EI is known as a continuous function of x it may be integrated twice to yield the slope ($dy/dx = i$) and the deflection (y) at any point.

Thus
$$i = \frac{dy}{dx} = \int \frac{M}{EI} dx + A \quad \text{Eq. 8.4}$$

$$y = \int \int \frac{M}{EI} dx dx + Ax + B \quad \text{Eq. 8.5}$$

where A and B are constants of integration determined by the end conditions.

The integration may be performed analytically, if M/EI is known as a continuous function of x , or graphically. The method will be illustrated by reference to standard types of loading, and general methods will be developed for any type of loading.

8.3. Sign Conventions

The sign convention adopted here for load, shear force and bending moment is that given in Volume I—i.e., *load* is *positive upwards*, *shear force* is *positive* when the summation of forces to the *left* of the point considered is *upwards*, and *bending moment* is *positive* when it produces *compression* in the *upper* surface of the beam. The additional convention now introduced is that *deflection* (y) is

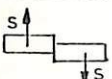
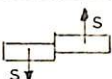
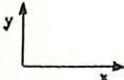
	POSITIVE	NEGATIVE
<u>LOAD</u> (w)		
<u>SHEAR</u> <u>FORCE</u> (s)		
<u>BENDING</u> <u>MOMENT</u> (M)		
<u>x and y</u>		
<u>SLOPE</u> (i)		

FIG. 8.1.—SIGN CONVENTIONS.

positive upwards, and that *distance* (x) *measured along the beam* from the origin is generally *positive to the right*. These conventions are illustrated in Fig. 8.1 together with that for slope. It should be noted that occasionally it may be convenient to reverse the convention for the sign of x , in which case the convention for the slope will also be reversed if y remains positive upwards.

8.4. Application of Basic Equation to Standard Types of Loading

CASE I—UNIFORM BEAM SUBJECTED TO CONSTANT BENDING MOMENT WITH ZERO SHEAR

This is the simplest case, one which can be solved geometrically. The loading, shown in Fig. 8.2, consists of a couple of magnitude M applied to each end. Take



FIG. 8.2.—BEAM SUBJECTED TO CONSTANT BENDING MOMENT WITH ZERO SHEAR.

A as the origin, and measure x positive to the right. Then at x from A, the bending moment is M .

From equation 8.3

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

Since the beam is uniform, EI is constant and we can integrate as follows

$$\frac{dy}{dx} = \frac{M}{EI}x + A \quad \text{Eq. 8.6}$$

where A is a constant; and

$$y = \frac{M}{EI} \cdot \frac{x^2}{2} + Ax + B \quad \text{Eq. 8.7}$$

where B is another constant.

At A , $x = 0$ and $y = 0$ so that from equation 8.7, $B = 0$.

At B , $x = L$, and $y = 0$ so that $A = -ML/2EI$.

Substituting these values of A and B in equations 8.6 and 8.7 we have that the slope at any point is given by

$$\frac{dy}{dx} = \frac{M}{EI} \left(x - \frac{L}{2} \right) \quad \text{Eq. 8.8}$$

and the deflection at any point is given by

$$y = \frac{M}{2EI} (x^2 - Lx) \quad \text{Eq. 8.9}$$

The maximum deflection will obviously occur at the centre where $x = L/2$ (this follows from symmetry or from the fact that $dy/dx = 0$ when $x = L/2$), and from equation 8.9 we obtain

$$y_{\max.} = \frac{M}{2EI} \left(\frac{L^2}{4} - \frac{L^2}{2} \right) = -\frac{ML^2}{8EI} \quad \text{Eq. 8.10}$$

Note that the negative sign agrees with our convention.

From equation 8.8, the slope at each end of the beam is numerically equal to $ML/2EI$, being positive at B and negative at A .

These results may be verified geometrically by noting that the radius of curvature, R , of the beam is constant, since $1/R = M/EI$, so that the beam bends into an arc of a circle, and by remembering that deflections are small compared with the length of the beam. In Fig. 8.3, the arc ACB represents the deflected

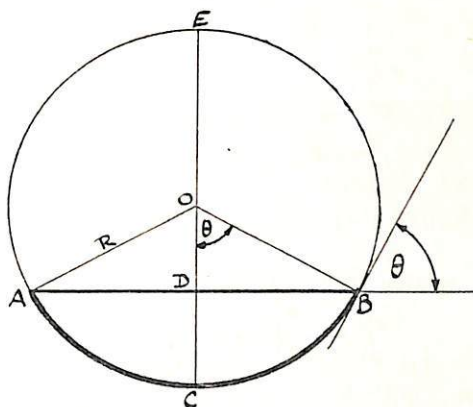


FIG. 8.3.—GEOMETRICAL DETERMINATION OF SLOPE AND DEFLECTION FOR CASE I.

form of the beam, the length ADB being L . O is the centre of the circle, radius R , of which ACB is an arc, and D is the mid-point of AB .

Then DC is the maximum deflection of the beam. From the geometrical properties of a circle it follows that

$$\begin{aligned} DC \times DE &= AD \times DB \\ DC(2R - DC) &= L^2/4. \end{aligned}$$

Since DC is assumed small compared with L and with R , we may neglect $(DC)^2$ in comparison with the other terms in this equation.

$$\begin{aligned} \text{Hence} \quad DC \times 2R &= \frac{L^2}{4} \\ \text{so that} \quad DC &= \frac{L^2}{8R} = \frac{ML^2}{8EI} \end{aligned}$$

(since $\frac{1}{R} = \frac{M}{EI}$) which agrees with equation 8.10.

$$\text{The slope at B} = i_B = \tan \theta = \frac{DB}{OD} = \frac{L/2}{R - DC} \approx \frac{L}{2R}$$

since DC is small compared with R .

Hence

$$i_B = \frac{ML}{2EI}$$

using $\frac{1}{R} = \frac{M}{EI}$, which again agrees with our previous result.

CASE II—CANTILEVER WITH CONCENTRATED LOAD AT TIP (Fig. 8.4)

Taking the origin, O, at the root, and measuring x positive to the right from O, the bending moment at distance x from O is given by

$$M_x = W(L - x)$$

$$\text{Hence} \quad \frac{d^2y}{dx^2} = \frac{W}{EI}(L - x) \quad \dots \dots \dots \text{Eq. 8.11}$$

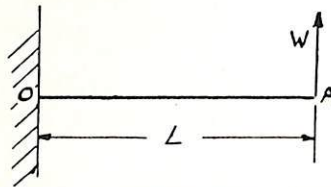


FIG. 8.4.—CANTILEVER WITH CONCENTRATED LOAD AT TIP (CASE II).

If EI is constant along the beam, this equation may be integrated to yield

$$\frac{dy}{dx} = \frac{W}{EI} \left(Lx - \frac{x^2}{2} \right) + A$$

where A is a constant. But when $x = 0$, $dy/dx = 0$ since the beam is "fixed" at O, so that $A = 0$

$$\frac{dy}{dx} = \frac{W}{EI} \left(Lx - \frac{x^2}{2} \right) \quad \dots \dots \dots \text{Eq. 8.12}$$

Equation 8.12 gives the slope at any point along the beam, and in particular the slope at the tip, i_A , where $x = L$, will be

$$i_A = \frac{WL^2}{2EI} \quad \text{Eq. 8.13}$$

Integrating equation 8.12 gives

$$y = \frac{W}{EI} \left(\frac{Lx^2}{2} - \frac{x^3}{6} \right) + B$$

where B is a constant.

At O , $x = 0$ and $y = 0$ so that $B = 0$. Hence

$$y = \frac{W}{EI} \left(\frac{Lx^2}{2} - \frac{x^3}{6} \right) \quad \text{Eq. 8.14}$$

gives the deflection at any point.

At the tip the deflection y_A is obtained by putting $x = L$ in equation 8.14, and

$$y_A = \frac{WL^3}{3EI} \quad \text{Eq. 8.15}$$

CASE IIA—CANTILEVER WITH CONCENTRATED LOAD INBOARD OF THE TIP (Fig. 8.5)

Equations 8.11 to 8.15 above will apply to the portion OB of the cantilever if we put $L = a$ in all these equations.

Hence

$$y_B = \frac{Wa^3}{3EI}$$

and

$$i_B = \frac{Wa^2}{2EI}$$

Between B and A there is no bending moment to produce curvature in the

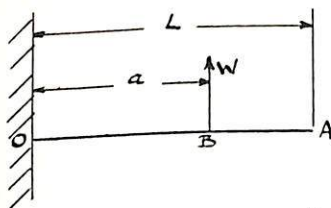


FIG. 8.5.—CANTILEVER WITH LOAD INBOARD OF TIP (CASE IIA).

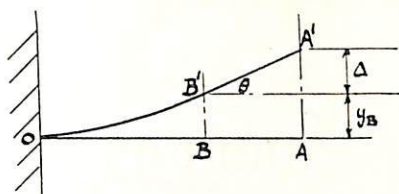


FIG. 8.6.—DEFLECTED FORM OF BEAM OF FIG. 8.5.

beam so that the portion BA will remain a straight line. Referring to Fig. 8.6, the deflection at A is

$$y_A = y_B + \Delta$$

Remembering that slopes and deflections are small, we can write with considerable accuracy

$$\Delta = (L - a)i_B$$

($\Delta = B'A' \sin \theta = (L - a) \sin \theta$ and for small values of θ , $\sin \theta = \tan \theta = i_B$).

Hence

$$y_A = y_B + (L - a)i_B$$

CASE III—CANTILEVER WITH UNIFORMLY DISTRIBUTED LOAD (Fig. 8.7)

With the same notation as in the previous case,

$$M_x = \frac{w(L-x)^2}{2} = \frac{w}{2}(L^2 - 2Lx + x^2)$$

$$\frac{d^2y}{dx^2} = \frac{w}{2EI}(L^2 - 2Lx + x^2)$$

On integrating we have

$$\frac{dy}{dx} = \frac{w}{2EI}\left(L^2x - Lx^2 + \frac{x^3}{3}\right) + A$$

where A is a constant.

As before, when $x = 0$, $dy/dx = 0$ so that $A = 0$ and the slope at any point is given by

$$\frac{dy}{dx} = \frac{w}{2EI}\left(L^2x - Lx^2 + \frac{x^3}{3}\right) \quad \text{Eq. 8.16}$$

The slope at the tip is

$$i_A = \frac{wL^3}{6EI} \quad \text{Eq. 8.17}$$

Integrating equation 8.16 and noting that when $x = 0$, $y = 0$ we obtain for the deflection at any point

$$y = \frac{w}{2EI}\left(\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right) \quad \text{Eq. 8.18}$$

At the tip, where $x = L$

$$y_A = \frac{wL^4}{8EI} \quad \text{Eq. 8.19}$$

For combinations of the above types of loading, the deflections due to each type of loading may be computed separately and these deflections may then be

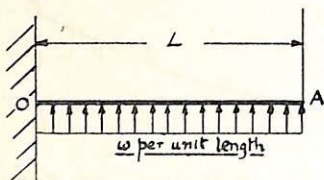


FIG. 8.7.—CANTILEVER WITH UNIFORMLY DISTRIBUTED LOAD (CASE III).

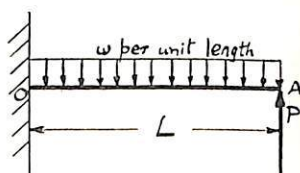


FIG. 8.8.—PROPPED CANTILEVER FOR EXAMPLE 8.1.

added to give the resultant deflection. A very useful application of this principle is illustrated by the following example:—

Example 8.1. Fig. 8.8 shows a cantilever carrying a uniformly distributed load of w per unit length, with the tip, A, propped to the level of the root. We are required to find the load, P , in the prop. This problem cannot be solved by the usual methods of statics, since there are three unknown quantities (the load P , the force and the fixing moment at the root) and only two equations of equilibrium from which to determine them. If, however, we consider the deflection

of the beam the problem becomes simple. As indicated in Fig. 8.9, the loading in the cantilever may be regarded as the sum of the distributed load and the load P acting separately.

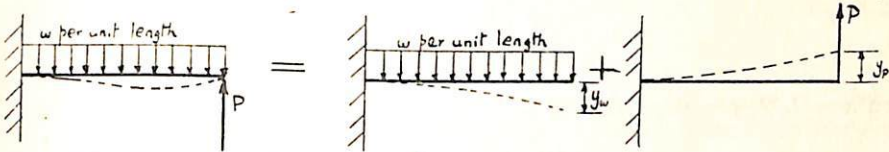


FIG. 8.9.—RESULTANT LOAD AND DEFLECTION AS SUM OF TWO SEPARATE LOADS AND DEFLECTIONS.

The distributed load alone will produce a deflection y_w downwards at the tip, and P alone will produce a deflection y_P upwards. If the tip is propped to the same level as the fixed end, the resultant tip deflection must be zero.

Hence $y_P = y_w$

Inserting values for y_P and y_w as given respectively by equations 8.15 and 8.19, this gives

$$\frac{PL^3}{3EI} = \frac{wL^4}{8EI}$$

so that

$$P = \frac{3}{8}wL$$

A similar analysis may be followed for a prop at any other point and for a prop which itself deflects under load.

CASE IV—SIMPLY SUPPORTED BEAM WITH CENTRAL LOAD (Fig. 8.10)

Since the beam is symmetrically loaded we need consider only one half of the beam, and we take our origin at O , the mid-point, and consider OB , x being measured positive to the right from O .

At x from O , $M_x = -\frac{W}{2}\left(\frac{L}{2} - x\right)$

i.e., $\frac{d^2y}{dx^2} = -\frac{W}{2EI}\left(\frac{L}{2} - x\right)$

Integrating $\frac{dy}{dx} = -\frac{W}{2EI}\left(\frac{L}{2}x - \frac{x^2}{2}\right) + A$

where A is a constant.

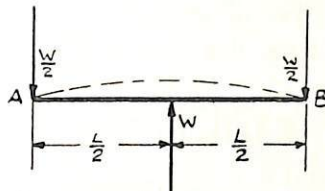


FIG. 8.10.—SIMPLY SUPPORTED BEAM WITH CENTRAL LOAD (CASE IV).

By symmetry $dy/dx = 0$ at O , when $x = 0$, so that $A = 0$. Then

$$\frac{dy}{dx} = -\frac{W}{2EI}\left(\frac{L}{2}x - \frac{x^2}{2}\right) \quad \text{Eq. 8.20}$$

and
$$i_B = -\frac{WL^2}{16EI}; i_A = +\frac{WL^2}{16EI} \quad \dots \quad \text{Eqs. 8.21}$$

Integrating equation 8.20 yields

$$y = -\frac{W}{2EI} \left(\frac{Lx^2}{4} - \frac{x^3}{6} \right) + B$$

When $x = L/2$, $y = 0$ so that $B = +\frac{WL^3}{48EI}$, and then

$$y = -\frac{W}{2EI} \left(\frac{Lx^2}{4} - \frac{x^3}{6} \right) + \frac{WL^3}{48EI} \quad \dots \quad \text{Eq. 8.22}$$

The maximum deflection is at O, where $x = 0$, and is given by

$$y_{\max.} = +\frac{WL^3}{48EI} \quad \dots \quad \text{Eq. 8.23}$$

CASE V—SIMPLY SUPPORTED BEAM WITH UNIFORMLY DISTRIBUTED LOAD (Fig. 8.11)

Taking the origin at A and measuring x positive to the right, we have

$$M_x = -\frac{wLx}{2} + \frac{wx^2}{2}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{w}{2EI}(Lx - x^2)$$

Integrating,
$$\frac{dy}{dx} = -\frac{w}{2EI} \left(\frac{Lx^2}{2} - \frac{x^3}{3} \right) + A$$

Integrating again,
$$y = -\frac{w}{2EI} \left(\frac{Lx^3}{6} - \frac{x^4}{12} \right) + Ax + B$$

When $x = 0$, $y = 0$ so that $B = 0$.

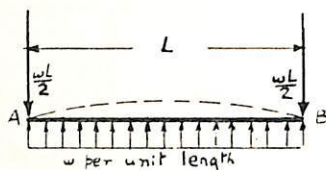


FIG. 8.11.—SIMPLY SUPPORTED BEAM WITH UNIFORMLY DISTRIBUTED LOAD (CASE V).

Also, when $x = L$, $y = 0$, so that $A = +\frac{wL^3}{24EI}$. Hence

$$\frac{dy}{dx} = -\frac{w}{2EI} \left(\frac{Lx^2}{2} - \frac{x^3}{3} - \frac{L^3}{12} \right) \quad \dots \quad \text{Eq. 8.24}$$

At the ends of the beam,

$$i_A = +\frac{wL^3}{24EI} \quad \text{and} \quad i_B = -\frac{wL^3}{24EI} \quad \dots \quad \text{Eqs. 8.25}$$

At any point,
$$y = -\frac{w}{2EI} \left(\frac{Lx^3}{6} - \frac{x^4}{12} - \frac{L^3x}{12} \right) \quad \dots \quad \text{Eq. 8.26}$$

Obviously the maximum deflection occurs at the centre of the beam (this may be verified by noting that $dy/dx = 0$ when $x = L/2$), and is given by substituting $x = L/2$ in equation 8.26.

Then
$$y_{\max.} = + \frac{5 w L^4}{384 EI} \quad \text{Eq. 8.27}$$

8.5. General Methods for any Loading

The results obtained above are all easy to remember and derive. There are several special methods of determining deflections, which have been developed

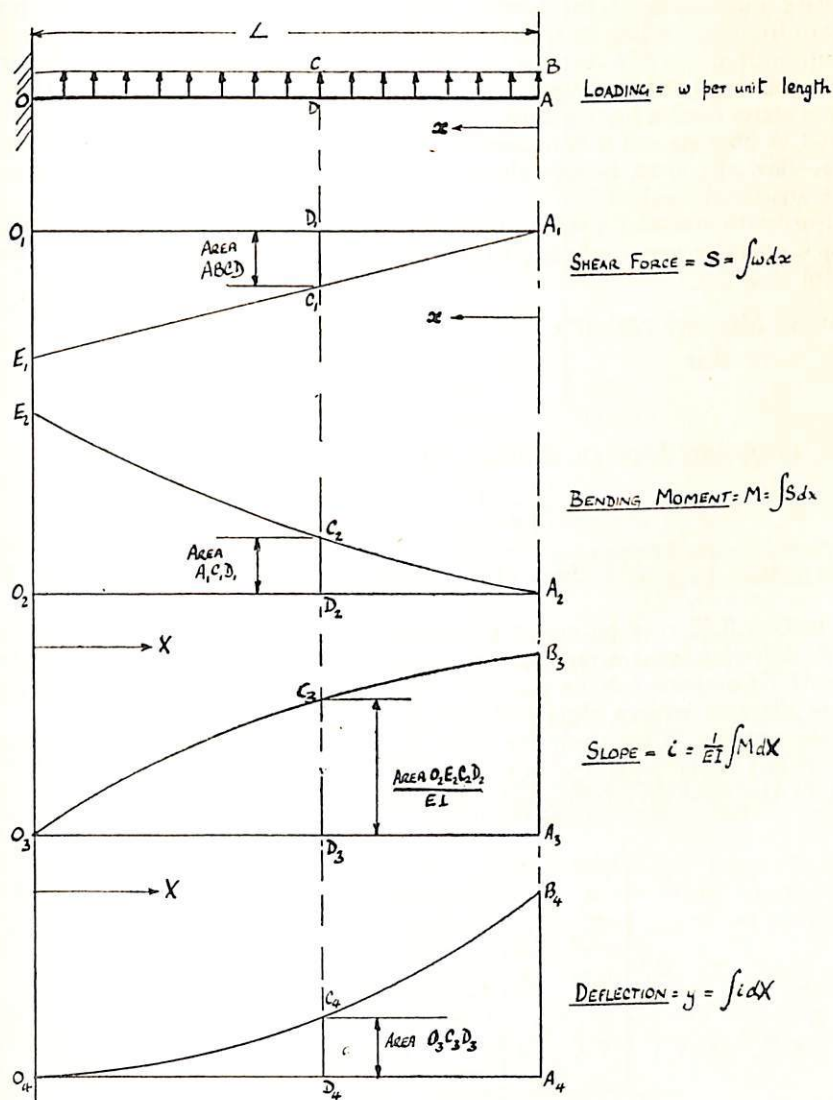


FIG. 8.12.—DETERMINATION OF SLOPE AND DEFLECTION BY GRAPHICAL INTEGRATION.

for certain types of loading. It is not proposed to discuss all these methods here, and we shall proceed to consider only some general semi-graphical methods for dealing with any type of loading.

(a) GRAPHICAL INTEGRATION

Equations 8.3, 8.4 and 8.5 indicate that slopes and deflections may be determined by successive graphical integrations of the bending moment diagram (or of the M/EI curve if EI is not constant along the beam). Remembering from Volume I that the shear force and bending moment diagrams can be derived by successive integration of the loading diagram, obviously the complete process may start from the loading diagram. A typical sequence of diagrams for a cantilever with upwards uniformly distributed load is shown in Fig. 8.12. Each curve is the integral (or $1/EI \times$ the integral) of the one preceding it. Note that each diagram starts from a known zero. In less-simple cases the points of zero slope may not be obvious and it is necessary to adopt some method of determining such points—this, of course, is equivalent to determining the constants of integration in the analytical method.

In order to arrive at a method suitable for dealing with any type of loading, we shall consider some relationships between slope, deflection and the bending moment diagram.

(b) AREA—MOMENT METHOD

We know that

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

Hence, integrating between limits x_1 and x_2 ,

$$\left[\frac{dy}{dx} \right]_{x_1}^{x_2} = \int_{x_1}^{x_2} \frac{M}{EI} dx$$

$$\text{i.e., } i_2 - i_1 = \int_{x_1}^{x_2} \frac{M}{EI} dx = \text{Area of } \frac{M}{EI} \text{ diagram between } x_1 \text{ and } x_2 \quad \text{Eq. 8.28}$$

Equation 8.28 may be stated in words as follows:—

The difference between the slopes at any two points on a beam is equal to the area of the M/EI diagram between these points (or to $1/EI$ times the area of the bending moment diagram between these points if EI is constant along the beam).

Consequently, if we know the slope at any one point we may determine the slope at any other point by application of equation 8.28.

With the notation of Fig. 8.13 we may write

$$i_2 - i_1 = A_{12}$$

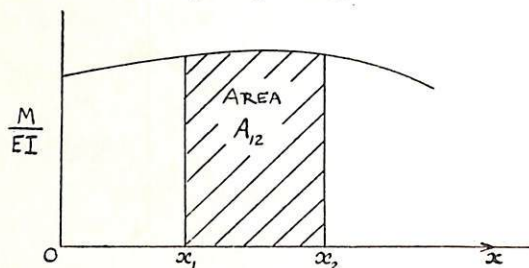


FIG. 8.13.—NOTATION FOR AREA—MOMENT METHOD.

Now, reverting to our basic equation, multiply both sides by x , giving

$$x \frac{d^2y}{dx^2} = \frac{M}{EI} x$$

Integrating the left-hand side of this equation by parts, between the limits x_1 and x_2 , yields

$$\left[x \frac{dy}{dx} - y \right]_{x_1}^{x_2} = \int_{x_1}^{x_2} \frac{M}{EI} x dx$$

or $(x_2 i_2 - y_2) - (x_1 i_1 - y_1) = \text{Moment of } A_{12} \text{ about the origin, O}$ Eq. 8.29

The physical meaning of the quantity $(xi - y)$ is simply the intercept of the tangent at x on the y -axis, as may be seen from Fig. 8.14. For positive values

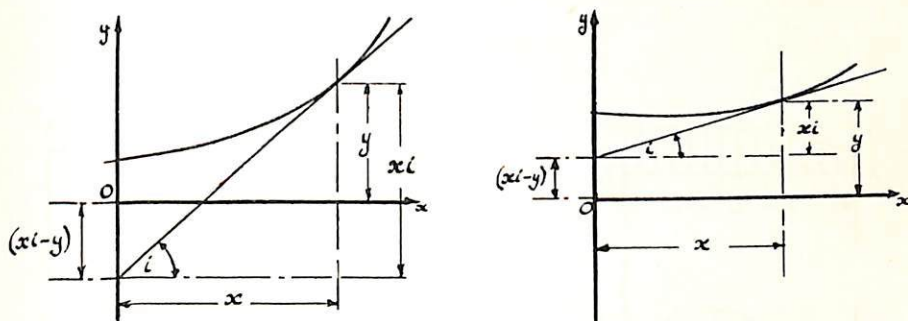


FIG. 8.14.—PHYSICAL INTERPRETATION OF $(xi - y)$.

of x , i and y , the intercept will be positive when y is greater than xi , so that we should state correctly

$$xi - y = -(\text{intercept on } y\text{-axis of tangent at } x).$$

If, now, we choose our limits x_1 and x_2 correctly it is possible from equations 8.28 and 8.29 to determine the quantities we require for complete solution of the deflection problem.

Suppose, first, that we take x_1 at the origin, so that x_1 is zero.

Then, let x_2 be at a point of zero slope, so that i_2 is zero.

Equation 8.29 becomes in this case

$$-y_2 + y_1 = \text{moment about origin of } A_{12} \quad \text{Eq. 8.30}$$

Hence, between the origin and a point of zero slope the difference between the deflections is equal to the moment of the M/EI diagram between these points about the origin. If either of these points is also a point of zero deflection, then the deflection at the other point is given directly.

Secondly, let us consider the part of a beam between two supports on the same level. Take one of these support points, x_1 , at the origin, so that x_1 is zero. Since the supports are on the same level, $(-y_2 + y_1)$ is zero, taking x_2 at the second support point. Then equation 8.29 becomes

$$x_2 i_2 = \text{moment about origin of } A_{12} \quad \text{Eq. 8.31}$$

Thus for a beam supported at A and B, with M/EI diagram as in Fig. 8.15, the area of this diagram being A and the centroid of the diagram being $\bar{x}$ from A, we determine the slope at B as

$$i_B = \frac{A\bar{x}}{L} \quad \text{Eq. 8.32}$$

Similarly, taking B as the origin we could write

$$i_A = - \frac{A(L - \bar{x})}{L} \quad \text{Eq. 8.33}$$

The negative sign for i_A arises out of the fact that taking B as the origin we are measuring x in the negative direction.

It may be noted that, with the exception of sign, the procedure for determining the end slopes for a beam simply supported at its ends is the same as that for determining the support reactions for loading of the form of the M/EI diagram.

Having determined the slope at a support point by this means it is an easy matter to apply equation 8.28 to give the slope at any other desired point. Equation 8.29 may then be used to give the deflection desired.

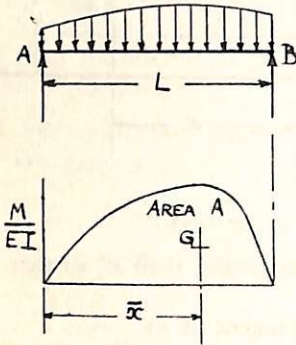


FIG. 8.15.—BEAM SUPPORTED AT TWO POINTS ON SAME LEVEL.

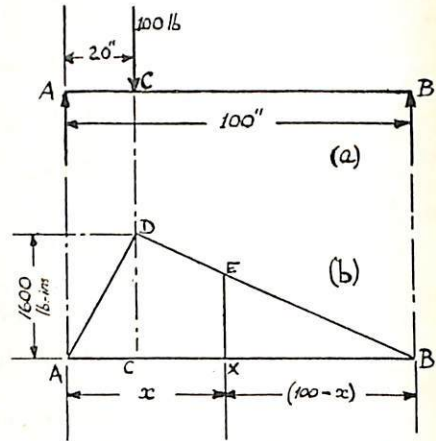


FIG. 8.16.—BEAM FOR EXAMPLE 8.2.

As an illustration, consider the beam in Fig. 8.15, supported at A and B and loaded so as to produce the M/EI diagram between A and X shown. Let A_x be the area of the M/EI diagram between A and X, and let $\bar{x}_1$ be the distance of the centroid of A_x from A.

i_A may be determined as just described.

Then from equation 8.28 $i_x = i_A + A_x \quad \text{Eq. 8.34}$

It should be remembered that A_x should be given its correct sign in this equation, as in all other equations in this section.

Remembering that, taking A as origin, $x_1 = y_1 = 0$ we may write from equation 8.28

$$xi_x - y_x = A_x \bar{x}_1$$

Hence

$$y_x = xi_x - A_x \bar{x}_1 \quad \text{Eq. 8.35}$$

Using equation 8.34 this becomes

$$\begin{aligned} y_x &= x(i_A + A_x) - A_x \bar{x}_1 \\ &= xi_A + A_x(x - \bar{x}_1) \end{aligned} \quad \text{Eq. 8.36}$$

Equations 8.35 or 8.36 may be used to obtain the value of y_x , whichever is

the more convenient. These equations may be expressed in words as follows :— The deflection at a point X distant x from a point of support, taken as the origin, is given by x times the slope at this point minus the moment about the origin of the area of the M/EI diagram between the origin and that point, or by x times the slope at the origin plus the moment about X of the area of the M/EI diagram from the origin to X.

The necessary areas and moments may be determined analytically or graphically.

Example 8.2. The uniform beam AB has $EI = 2.4 \times 10^6$ lb. in.² Determine the deflection at C, the point of loading, and obtain also the position and magnitude of maximum deflection. The bending moment diagram is as indicated in Fig. 8.16. Since EI is constant, we may use everywhere $1/EI$ times the area of the bending moment diagram instead of the area of the M/EI diagram.

Applying equation 8.32,

$$\begin{aligned} i_B &= \frac{1}{100} \times \frac{\text{Moment of area ADB about A}}{EI} \\ &= \frac{1}{100EI} (\text{Moment of ADC about A} + \text{Moment of CDB about A}) \\ &= \frac{1}{100EI} \left(\frac{1}{2} \times 1600 \times 20 \times \frac{2}{3} \times 20 + \frac{1}{2} \times 1600 \times 80 \left(20 + \frac{80}{3} \right) \right) \\ &= \frac{1}{300EI} (1600 \times 40(10 + 140)) \\ &= \frac{640 \times 150}{3EI} \end{aligned}$$

$$\begin{aligned} i_A &= -\frac{1}{100} \times \frac{\text{Moment of area ADB about B}}{EI} \\ &= -\frac{1}{100EI} \left[\frac{1}{2} \times 1600 \times 20 \left(80 + \frac{20}{3} \right) + \frac{1}{2} \times 1600 \times 80 \times \frac{2}{3} \times 80 \right] \\ &= -\frac{1600 \times 40}{300EI} (65 + 160) = -\frac{640 \times 225}{3EI} \end{aligned}$$

(Note : equation 8.34 gives

$$\begin{aligned} i_B &= i_A + \frac{\text{Area ADB}}{EI} \\ \text{i.e., } i_A &= \frac{640 \times 150}{3EI} - \frac{1}{2} \times \frac{1600 \times 100}{EI} \\ &= -\frac{640 \times 225}{3EI}, \text{ which agrees with the above result.} \end{aligned}$$

We can work from either A or B, and in order to avoid any confusion with signs it is probably preferable to work from A in the positive direction of x .

Now, to find y_C , we apply equation 8.36.

$$\begin{aligned} y_C &= AC \times i_A + \frac{\text{Moment of area ADC about C}}{EI} \\ &= 20 \left(-\frac{640 \times 225}{3EI} \right) + \frac{1}{2} \times \frac{1600 \times 20 \times 20/3}{EI} \\ &= -\frac{640 \times 4000}{3EI} \end{aligned}$$

In order to determine the position of maximum deflection we need to know the position of zero slope. At x from A we have

$$i_x = i_A + \frac{\text{Area of B.M. diagram from A to } x}{EI}$$

and if i_x is to be zero, we must have

$$\frac{\text{Area of B.M. diagram from A to } x}{EI} = -i_A = -\left(-\frac{640 \times 225}{3EI}\right)$$

Hence, the area of the B.M. diagram from A to $x = \frac{640 \times 225}{3}$

Since the area ADC is less than this amount, the position of zero slope and maximum deflection will obviously lie between C and B. For simplicity, therefore, it will be easier to work between x and B, as the area involved is then only that of a triangle. Equation 8.28 gives

$$i_B = i_x + \frac{\text{Area BEX}}{EI} \quad (\text{see Fig. 8.16 (b)})$$

If i_x is to be zero, this leads to

$$\text{Area BEX} = EI i_B$$

$$\text{Hence,} \quad \frac{1}{2} \times \frac{1600(100 - x)^2}{80} = \frac{640 \times 150}{3}$$

$$\text{i.e.,} \quad (100 - x) = 56.56 \text{ in.}$$

Hence the point of zero slope is 56.56 in. from B, or 43.44 in. from A, and 23.44 in. from C. The deflection at this point may be found by application of equation 8.35, which, since i_x is zero, reduces to

$$y_x = - \frac{\text{Moment of area ADEX about A}}{EI}$$

We may simplify our calculations by taking B as origin, so that equation 8.35, with i_x zero, becomes

$$\begin{aligned} y_x &= - \frac{\text{Moment of area BEX about B}}{EI} \\ &= - \frac{1}{2EI} \left(\frac{1600 \times 56.56}{80} \times 56.56 \right) \times \frac{2}{3} \times 56.56 \\ &= - \frac{640 \times 100 \times 56.56}{3EI} \text{ in.} \end{aligned}$$

Inserting the value of $EI = 2.4 \times 10^6$ we obtain the required results as follows :—

$$y_C = -0.355 \text{ in.}$$

Max. deflection = 0.503 in. at 43.44 in. from A.

The above example illustrates the general procedure, although with the simple loading given, the calculation of the necessary areas and moments of areas is an easy matter. In order to show how the general method may be applied for more difficult loadings, the following example is presented :—

Example 8.3. For the beam in Fig. 8.17, simply supported at A and B, the ordinates of the loading diagram are given in the following table:—

Distance from A (in.)	0	5	10	15	20	25	30	
Load intensity (lb. per in.)	- 5	- 6.35	- 7.0	- 7.0	- 6.35	- 5.0	- 3.0	(all negative)

Determine the deflection curve for the beam.

The calculations are set out in tabular form, and the following brief explanation will serve to clarify the various steps.

Cols. 3 and 4 comprise the integration of the loading diagram, the figures in col. 4 giving the total load between A and the point considered, so that col. 4 actually gives the shear force due to the distributed loading alone. Cols. 5 and 6

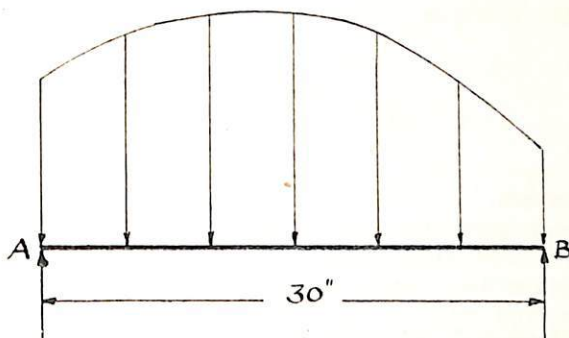


FIG. 8.17.—LOADING DIAGRAM FOR EXAMPLE 8.3.

comprise the integration of this shear force to give, in col. 6, the bending moment due to the distributed load alone. Thus, the final figure in col. 6 gives the moment about B of the loading on the beam, and the reaction at A, R_A , is obtained by dividing this figure by the distance between the supports. Algebraic addition of R_A to the figures in col. 4 yields col. 7, which is the resultant shear force along the beam. Col. 8 gives the moment of R_A about the various points along the beam, and the difference between these figures and those in col. 6 gives the resultant bending moment along the beam. Integration of the B.M. diagram, in cols. 10 and 11, gives the area of the B.M. diagram, and further integration of col. 11 in cols. 12 and 13 gives the moment of area of the B.M. diagram. Equation 8.33 indicates that the slope at A may be determined by dividing the last figure in col. 13 by EI times the distance between A and B. Col. 14 is then obtained by adding algebraically EI times i_A to the figures in col. 11. It would be possible now to derive the deflection curve by double integration of col. 14, but application of equation 8.36 gives a more convenient calculation. The figures in col. 13 represent, in the notation of equation 8.36, the quantity $EI A_x(x - \bar{x}_1)$, so that if we evaluate the product $EI i_A x$ and add algebraically to the figures in col. 13, we shall obtain $EI y$. This is done in cols. 15 and 16.

The slope and deflection curves may be obtained by plotting the figures in cols. 14 and 16 respectively against x .

x in. (1)	w lb./in. (2)	δW lb. (3)	W lb. (4)	δM_w lb. in. (5)	M_w lb. in. (6)	S lb. (7)	$R_A \cdot x$ lb. in. (8)
0	- 5.00	0.00	0.00	0.0	0.0	{ 0.00 94.94	0.0
2.5	- 5.85	- 13.57	- 13.57	- 16.9	- 16.9	81.37	237.4
5.0	- 6.35	- 15.24	- 28.81	- 53.0	- 69.9	66.13	474.7
7.5	- 6.85	- 16.50	- 45.31	- 92.6	- 162.5	49.63	711.8
10.0	- 7.00	- 17.31	- 62.62	- 134.9	- 297.4	32.32	949.2
12.5	- 7.10	- 17.62	- 80.24	- 178.5	- 475.9	14.70	1186.5
15.0	- 7.00	- 17.62	- 97.86	- 222.5	- 698.4	- 2.92	1423.9
17.5	- 6.85	- 17.31	- 115.17	- 266.2	- 964.6	- 20.23	1661.2
20.0	- 6.35	- 16.50	- 131.67	- 308.6	- 1273.2	- 36.73	1898.6
22.5	- 5.85	- 15.24	- 146.91	- 348.2	- 1621.4	- 51.97	2135.9
25.0	- 5.00	- 13.57	- 160.48	- 384.2	- 2005.6	- 65.54	2373.3
27.5	- 3.95	- 11.18	- 171.66	- 415.2	- 2420.8	- 76.72	2610.6
30.0	- 3.00	- 8.68	- 180.34	- 427.5	- 2848.3	{ - 85.40 0.00	2848.0

$$R_A = \frac{2848.3}{30.0} = 94.94 \text{ lb.}$$

8.6. Shear Deflection

For most beams, shear deflection is negligible in comparison with bending deflection. This is fortunate, because except in certain simple cases the determination of deflection due to shear is not an easy problem. The simplest case is that of the thin web beam, in which we assume the shear flow to be constant over the depth of the web, with the flanges alone contributing to the bending strength. Over a length δx of a web of thickness t and depth h , in which the shear flow is s , we know that the shear stress, q , is equal to s/t , and that the shear force, S , is equal to sh . The deformation of the web due to shear will cause strain energy to be stored in the web, equal in amount to the work done by the shear force, S , in moving through a distance δy_s , this distance being the deflection, due to shear, of the elementary length δx of the web. The cross-sectional area of the web, A , is ht , and from equation 2.2, the shear strain energy may be written as

$$U_s = \frac{S^2 \delta x}{2GA}$$

The work done by the shear force, S , in moving through a distance δy_s , assuming that S is applied gradually, will be $\frac{1}{2} S \delta y_s$, so that

$$\frac{1}{2} S \delta y_s = \frac{S^2 \delta x}{2GA}$$

i.e.,

$$\delta y_s = \frac{S \delta x}{GA} = \frac{s \delta x}{Gt} = \frac{q \delta x}{G}$$

Over a length L , the total shear deflection will be

$$y_s = \Sigma \delta y_s = \int_0^L \frac{s \delta x}{Gt} = \int_0^L \frac{q dx}{G}$$

In the case of a solid section beam we have seen that the shear stress is not

M lb. in. (9)	δA lb. in. ² (10)	$\Sigma \delta A$ lb. in. ² (11)	δa lb. in. ³ (12)	$\Sigma \delta a$ lb. in. ³ (13)	EI lb. in. ² (14)	$\alpha \cdot EI_A$ lb. in. ³ (15)	EIy lb. in. ³ (16)
0.0	0.0	0	0	0	- 7180	0	0
220.5	276.0	276	345	345	- 6904	- 17,950	- 17,605
404.8	782.0	1058	1668	2013	- 6122	- 35,900	- 33,887
549.3	1193.0	2251	4136	6149	- 4929	- 53,850	- 47,701
651.8	1501.0	3752	7504	13653	- 3428	- 71,800	- 58,147
710.6	1703.0	5455	11509	25162	- 1725	- 89,750	- 64,588
725.5	1795.0	7250	15881	41043	+ 70	- 107,700	- 66,657
696.6	1790.0	9040	20362	61405	1860	- 125,650	- 64,245
625.4	1665.0	10705	24681	86086	3525	- 143,600	- 57,514
514.5	1425.0	12130	28544	114630	4950	- 161,550	- 46,920
367.7	1103.0	13233	31704	146334	6053	- 179,500	- 33,166
189.8	697.0	13930	33954	180288	6750	- 197,450	- 17,162
0.0	237.0	14167	35121	215409	6987	- 215,400	0

$$EI_A = - \frac{215409}{30} = - 7180 \text{ lb. in.}^2$$

uniformly distributed over the cross-section, and account must be taken of this variation. An approximate determination for a rectangular section may be made as follows:—It should be noted at the outset that this method is not completely accurate, since it does not contain a complete investigation of the strain energy, assuming, as it does, that the ends of the beam element remain plane. It is, however, sufficient to give an indication of the order of the shear deflection.

In Fig. 8.18, ABCD is an element, of length δL , of a beam of rectangular

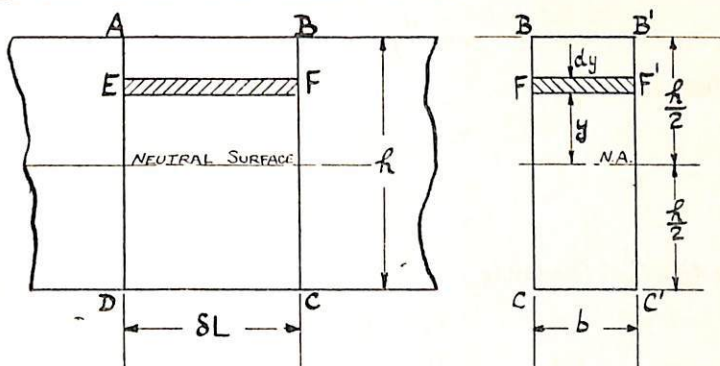


FIG. 8.18.—ELEMENT OF BEAM OF RECTANGULAR SECTION.

cross-section. Over this element of length the shear force, S , is assumed constant. Consider now the variation of shear stress over the cross-section. From para. 5.3, we know that at y from the neutral axis the shear stress is given by

$$q = \frac{6S}{bh^3} \left\{ \left(\frac{h}{2} \right)^2 - y^2 \right\}$$

The shear force on an element FF' of the cross-section is $qb dy$ and the strain

energy of an elementary slice, EF, of the beam will be

$$\delta U = \frac{(qb\delta y)^2 \delta L}{2Gb\delta y} = \frac{q^2 b \delta y}{2G} \cdot \delta L$$

Hence for the whole element ABCD the strain energy is given by

$$U = \frac{\delta L \cdot b}{2G} \int_{-h/2}^{h/2} q^2 dy$$

On substituting for q and integrating we obtain

$$U = \frac{3}{5} \cdot \frac{S^2}{Gb h} \cdot \delta L$$

Equating this to the work done by the shear force, S , in moving through a distance y_s , we determine

$$\frac{1}{2} S y_s = \frac{3}{5} \cdot \frac{S^2 \delta L}{Gb h}$$

or

$$y_s = \frac{6}{5} \cdot \frac{S}{Gb h} \cdot \delta L$$

For a beam of length L the shear deflection would then be given by a summation of the deflections for all elements such as the one considered.

Let us see now how this deflection compares in magnitude with that due to bending. For simplicity, consider a cantilever of length L , with a concentrated load at the tip. For a rectangular section, of breadth b and depth h , the second moment of area about the neutral axis is

$$I = \frac{bh^3}{12}$$

Then, if the applied load at the tip is W , the bending deflection is

$$y_b = \frac{WL^3}{3EI} = \frac{4WL^3}{Eb h^3}$$

and the shear deflection is

$$y_s = \frac{6WL}{5Gb h}$$

Therefore,

$$\frac{y_s}{y_b} = \frac{3}{10} \cdot \frac{E}{G} \cdot \left(\frac{h}{L}\right)^2$$

If we take G as $\frac{2}{5}E$ (assuming Poisson's ratio to be $1/4$) this becomes

$$\frac{y_s}{y_b} = \frac{3}{4} \left(\frac{h}{L}\right)^2$$

It follows that for a beam whose depth is small compared with its length, the shear deflection will be very small compared with the bending deflection, and in such a case it may generally be neglected.

BUILT-IN AND CONTINUOUS BEAMS

9.1. Previously we have considered beams with either two simple supports only or one fixed support only. Beams which are built-in at both ends, or are supported at more than two points, with either simple or fixed supports, are of frequent occurrence, but cannot be analysed by the methods of statics used so far. Several methods exist for dealing with these problems, and in this chapter we shall discuss methods based on deflections and slopes as well as on the equilibrium of the forces involved.

9.2. Beam built-in at Both Ends—General Problem

A beam built-in at both ends will, if the fixation is complete, have zero slope at each end as well as zero deflection. The loadings and the support reactions may be split into the two parts shown in Fig. 9.1 (c), and the bending moment

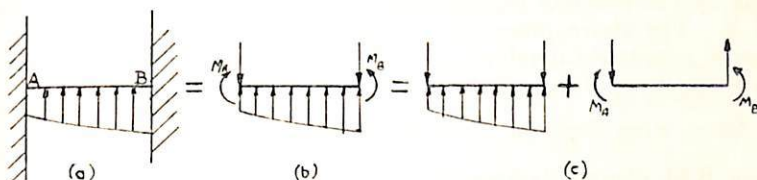


FIG. 9.1.—BUILT-IN BEAM.

diagram may be split into two corresponding parts, one of which, the “free moment diagram”, is that which would be obtained if the beam were simply supported, the other being that due to the fixing moments at the supports (Fig. 9.2).

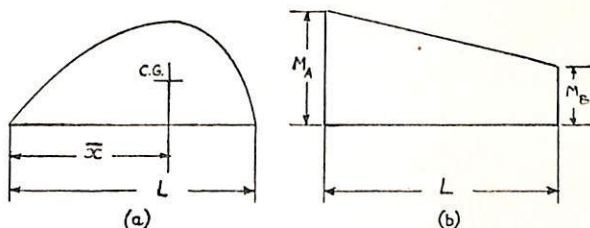


FIG. 9.2.—(a) FREE MOMENT DIAGRAM ; (b) FIXING MOMENT DIAGRAM.

Let the area of the free bending moment diagram be A and let the centroid of this area be at $\bar{x}$ from A. We assume in the first place that the fixing moments M_A and M_B are positive. The correct sign of the free moment diagram will, of course, be known (for the loading in Fig. 9.1 the free moment is negative).

From equation 8.34 we know that

$$i_B = i_A + \frac{\text{Area of B.M. diagram between A and B}}{EI}$$

But $i_B = i_A = 0$, so that the area of the resultant B.M. diagram between A and B must be zero

i.e.,
$$\left(\frac{M_A + M_B}{2}\right).L + A = 0 \quad \dots \quad \text{Eq. 9.1}$$

Furthermore, from equation 8.32, the moment of the resultant B.M. diagram about A is zero

i.e.,
$$M_B.L.\frac{L}{2} + \frac{1}{2}.(M_A - M_B).L.\frac{1}{3}.L + A.\bar{x} = 0$$

i.e.,
$$\left(\frac{M_A + 2M_B}{6}\right).L^2 + A.\bar{x} = 0 \quad \dots \quad \text{Eq. 9.2}$$

Equations 9.1 and 9.2 are sufficient to determine the fixing moments.

The reaction at the supports may be obtained by addition of the "free" reaction and the "fixing" reactions as under:—

$R_A = R'_A + \frac{M_A - M_B}{L}$ where R'_A = reaction at A assuming simple supports at A and B ("free" reaction at A).

$R_B = R'_B + \frac{M_B - M_A}{L}$ where R'_B = "free" reaction at B.

This problem could be solved also by integration of the differential equation for bending, and insertion of the boundary conditions of zero slope and deflection at A and B. The above procedure is a simplification of this method, making use of results previously obtained.

As an illustration, two special cases are considered below.

BUILT-IN BEAM WITH CENTRAL CONCENTRATED LOAD (Fig. 9.3)

The free B.M. diagram is negative, and its area, A , is $-\frac{1}{2} \cdot \frac{WL}{4} \cdot L = -\frac{WL^2}{8}$.

The centroid of the free B.M. diagram is at mid-span. Equation 9.1 above becomes

$$\frac{(M_A + M_B)L}{2} - \frac{WL^2}{8} = 0$$

i.e.,
$$M_A + M_B = \frac{WL}{4} \quad \dots \quad \text{Eq. 9.3}$$

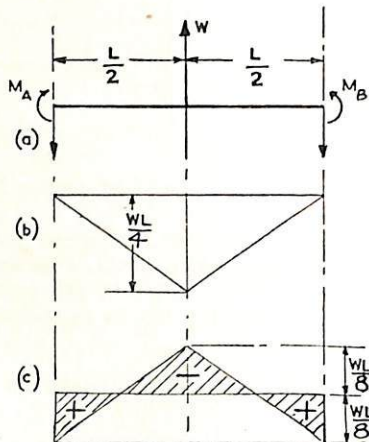


FIG. 9.3.—(a) BUILT-IN BEAM WITH CENTRAL CONCENTRATED LOAD; (b) FREE MOMENT DIAGRAM; (c) RESULTANT BENDING MOMENT DIAGRAM.

Equation 9.2 is

$$\frac{(M_A + 2M_B)L^2}{6} - \frac{WL^2}{8} \cdot \frac{L}{2} = 0$$

i.e.,

$$M_A + 2M_B = \frac{3WL}{8} \quad \text{Eq. 9.4}$$

From equations 9.3 and 9.4

$$M_A = \frac{WL}{8}; \quad M_B = \frac{WL}{8} \quad \text{Eq. 9.5}$$

Since the free and fixing moment diagrams are of opposite sign, we may obtain the algebraic sum of them by plotting both diagrams on the same side of the base line as in Fig. 9.3 (c), when the resultant diagram will be the shaded portion of this figure.

The reactions are as follows :—

$$R_A = \frac{W}{2} + \frac{M_A - M_B}{L} = \frac{W}{2} \text{ since } M_A = M_B$$

$$R_B = \frac{W}{2} + \frac{M_B - M_A}{L} = \frac{W}{2}.$$

BUILT-IN BEAM WITH UNIFORMLY DISTRIBUTED LOAD (Fig. 9.4)

Again the free moment diagram is negative. The area of the parabolic free moment diagram is $-\frac{wL^3}{12}$.

It is obvious in this case, as in the previous one, that by symmetry, $M_A = M_B$.

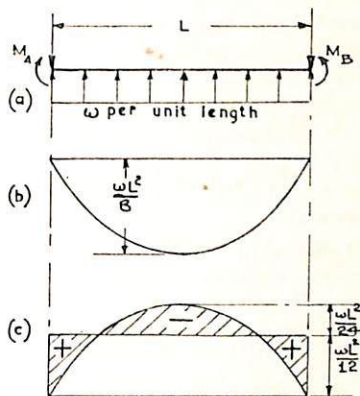


FIG. 9.4.—(a) BUILT-IN BEAM WITH UNIFORMLY DISTRIBUTED LOAD; (b) FREE MOMENT DIAGRAM; (c) RESULTANT BENDING MOMENT DIAGRAM.

Instead of using both equations 9.1 and 9.2, therefore, we may write equation 9.1 immediately as

$$\frac{2 \cdot M_A \cdot L}{2} - \frac{w \cdot L^3}{12} = 0$$

i.e.,

$$M_A = \frac{w \cdot L^2}{12} = M_B \quad \text{Eq. 9.6}$$

The resultant B.M. diagram is as shown in the figure, the B.M. at the centre being reduced in magnitude by the fixation of the ends from $w.L^2/8$ to $w.L^2/24$.

For other types of loading the procedure is essentially the same, although determination of areas and moments of areas of free bending moment diagrams may sometimes necessitate graphical integration. A simple numerical example for a non-central load follows.

Example 9.1. The beam shown in Fig. 9.5 (a) is built-in at both ends. Determine the fixing moments and the reactions at the supports, and draw the bending moment diagram.

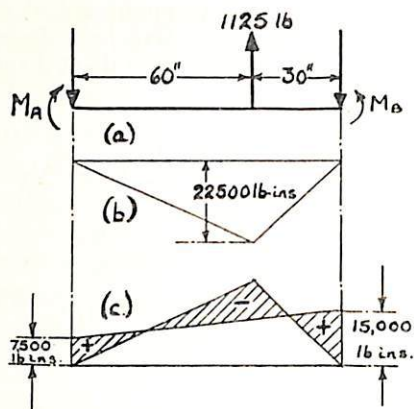


FIG. 9.5.

The free bending moment diagram is shown in Fig. 9.5 (b), the maximum value being 22,500 lb. in. (negative).

Inserting the appropriate numerical values, equation 9.1 becomes

$$(M_A + M_B) \times \frac{90}{2} - 22,500 \times \frac{90}{2} = 0$$

$$\text{i.e.,} \quad M_A + M_B = 22,500 \quad \dots \dots \dots (1)$$

Similarly, equation 9.2 gives

$$(M_A + 2M_B) \times \frac{90^2}{6} - \left[\frac{22,500 \times 60}{2} \times \frac{2}{3} \times 60 + \frac{22,500 \times 30}{2} \left(60 + \frac{30}{3} \right) \right] = 0$$

$$\text{i.e.,} \quad M_A + 2M_B = 22,500 \times \frac{5}{3} \quad \dots \dots \dots (2)$$

From (1) and (2) we obtain,

$$M_B = 15,000 \text{ lb. in.}$$

$$M_A = 7,500 \text{ lb. in.}$$

Superimposing the fixing moment diagram on the free moment diagram, both diagrams being drawn on the same side of the base line, the resultant bending moment diagram is as shown in Fig. 9.5 (c).

The reactions are calculated as follows:—

$$R_A = 1125 \times \frac{30}{90} + \frac{7500 - 15,000}{90} = 291.67 \text{ lb.}$$

$$R_B = 1125 \times \frac{60}{90} + \frac{15,000 - 7500}{90} = 833.33 \text{ lb.}$$

9.3. Continuous Beams

By this type of beam we understand a beam which is continuous over more than two supports. In the first case we shall consider, assume the intermediate supports to be simple supports, whilst the outermost supports may be simple or fixed. The most common means of determining the bending moment diagram is by using the *Three Moment Equation* (often referred to as Clapeyron's Equation of Three Moments), which can be derived as follows:—

In Fig. 9.6 let ABC be a beam supported at A, B, C, with the supports at A and C applying fixing moments M_A and M_C respectively, assumed positive. The product EI will be assumed constant in each of the two bays AB and BC, having the value E_1I_1 in the bay AB and E_2I_2 in the bay BC. For generality

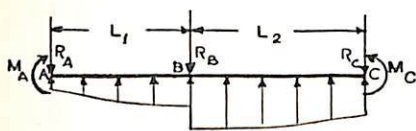
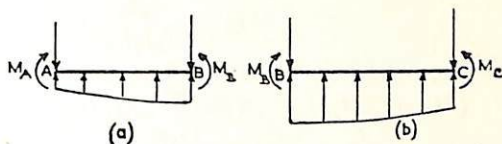


FIG. 9.6.

FIG. 9.7.—CONTINUOUS BEAM OF FIG. 9.6
RESOLVED INTO TWO BEAMS.

let the supports A, B, C be not collinear, A and C being displaced below B by amounts y_A and y_C respectively. The applied loading is assumed positive.

The effect of the portion BC on the portion AB is to apply a shear force and a moment at B, so that the bay AB could be isolated and loaded as shown in Fig. 9.7 (a), and similarly BC can be represented as in Fig. 9.7 (b).

As described in para. 9.2, the bending moment diagram for each of these bays may be divided into a free moment diagram and a fixing moment diagram, the resultant bending moment at any point being the algebraic sum of the free bending moment and the fixing bending moment.

Let the area of the free moment diagram for AB be A_1 , and let its centroid be at distance $\bar{x}_1$ from the end A; similarly let A_2 be the area of the free moment diagram for BC with its centroid at $\bar{x}_2$ from C. In order to develop the required equations, A_1 and A_2 will be assumed positive, although for the loading shown above they will be negative. If, however, we use positive signs for our symbolic expressions, the correct signs of unknown quantities will result when the correct signs of known quantities are inserted in the symbolic relationship.

The free moment and fixing moment diagrams are shown in Fig. 9.8. For the bay AB, take A as origin, measuring x positive to the right. Then, from equation 8.29, we obtain,

$$L_1 i_B - y_A = \frac{A_1 \bar{x}_1}{E_1 I_1} + \frac{1}{E_1 I_1} \left(\frac{M_B L_1^2}{2} + \frac{M_A - M_B}{2} \cdot L_1 \cdot \frac{L_1}{3} \right)$$

$$\text{or} \quad i_B = \frac{A_1 \bar{x}_1}{E_1 I_1 L_1} + \frac{y_A}{L_1} + \frac{(M_A + 2M_B)L_1}{6E_1 I_1} \quad \text{Eq. 9.7}$$

For the bay BC, take C as origin and measure x positive to the left. The slope at B will be the same as for the bay AB since the beam is continuous over the support B, but as we are measuring x positive in the opposite direction, the sign

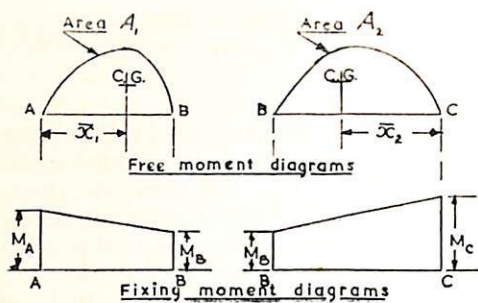


FIG. 9.8.—FREE AND FIXING MOMENT DIAGRAMS FOR TWO BAYS OF A CONTINUOUS BEAM.

of this slope will be changed. Hence we obtain

$$-i_B = \frac{A_2 \bar{x}_2}{E_2 I_2 L_2} + \frac{y_C}{L_2} + \frac{(M_C + 2M_B)L_2}{6E_2 I_2} \quad \text{Eq. 9.8}$$

Adding equations 9.7 and 9.8 to eliminate i_B we derive

$$\frac{(M_A + 2M_B)L_1}{6E_1 I_1} + \frac{(M_C + 2M_B)L_2}{6E_2 I_2} + \frac{A_1 \bar{x}_1}{E_1 I_1 L_1} + \frac{A_2 \bar{x}_2}{E_2 I_2 L_2} + \frac{y_A}{L_1} + \frac{y_C}{L_2} = 0$$

Rewriting, this becomes,

$$\begin{aligned} \frac{M_A L_1}{E_1 I_1} + 2M_B \left(\frac{L_1}{E_1 I_1} + \frac{L_2}{E_2 I_2} \right) + \frac{M_C L_2}{E_2 I_2} \\ + \frac{6A_1 \bar{x}_1}{E_1 I_1 L_1} + \frac{6A_2 \bar{x}_2}{E_2 I_2 L_2} + 6 \left(\frac{y_A}{L_1} + \frac{y_C}{L_2} \right) = 0 \quad \text{Eq. 9.9} \end{aligned}$$

This is the general form of the three moment equation for any type of lateral loading. Special cases, which are of frequent use, are as follows:—

For a beam which is uniform throughout its length $E_1 I_1 = E_2 I_2 = EI$ and the equation becomes

$$\begin{aligned} M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 \\ + \frac{6A_1 \bar{x}_1}{L_1} + \frac{6A_2 \bar{x}_2}{L_2} + 6EI \left(\frac{y_A}{L_1} + \frac{y_C}{L_2} \right) = 0 \quad \text{Eq. 9.10} \end{aligned}$$

If now the supports are collinear, $y_A = y_C = 0$ and we have a further simplification to

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 + \frac{6A_1 \bar{x}_1}{L_1} + \frac{6A_2 \bar{x}_2}{L_2} = 0 \quad \text{Eq. 9.11}$$

For certain types of loading, the term $\frac{6A\bar{x}}{L}$ takes simple forms which can be calculated quite easily.

For a *uniformly distributed load* of magnitude w per unit length, the free bending moment at x from A (Fig. 9.9) will be

$$M = -\frac{wL}{2} \cdot x + \frac{wx^2}{2}$$

The moment of the free bending moment diagram about A is

$$A\bar{x} = \int_0^L M \cdot x \cdot dx$$

$$\begin{aligned}
 &= \int_0^L \left(-\frac{wLx^2}{2} + \frac{wx^3}{2} \right) dx \\
 &= \left[-\frac{wLx^3}{6} + \frac{wx^4}{8} \right]_0^L = -\frac{wL^4}{24}
 \end{aligned}$$

Hence in this case,

$$\frac{6A\bar{x}}{L} = -\frac{wL^3}{4}$$

(Note that w is here assumed positive, i.e., upwards.) Consequently for a

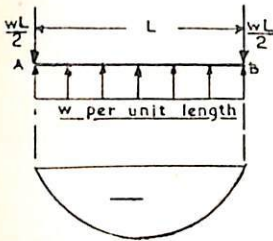


FIG. 9.9.—FREE MOMENT DIAGRAM FOR UNIFORMLY DISTRIBUTED LOAD.

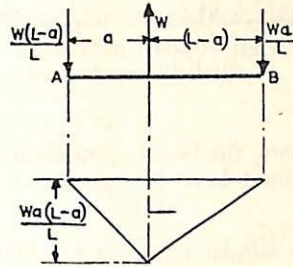


FIG. 9.10.—FREE MOMENT DIAGRAM FOR CONCENTRATED LOAD.

uniformly distributed load of magnitude w_1 per unit length in the bay AB and w_2 per unit length in the bay BC, equation 9.11 becomes

$$M_A L_1 + 2M_B(L_1 + L_2) + M_C L_2 - \frac{w_1 L_1^3}{4} - \frac{w_2 L_2^3}{4} = 0 \quad \text{Eq. 9.12}$$

For a *concentrated load* W , at distance a from the end A, the free bending moment diagram is as shown in Fig. 9.10, the moment being negative.

The moment of the area of this diagram about A is

$$\begin{aligned}
 -A\bar{x} &= \left(\frac{1}{2} \cdot \frac{W(L-a)a}{L} \times a \right) \cdot \frac{2a}{3} + \left(\frac{1}{2} \cdot \frac{W(L-a)a}{L} (L-a) \right) \left(a + \frac{L-a}{3} \right) \\
 &= \frac{W(L-a)a}{6L} (2a^2 + (L-a)(L+2a)) \\
 &= \frac{W(L-a)a}{6L} (L^2 + aL) \\
 \therefore A\bar{x} &= -\frac{Wa}{6} (L^2 - a^2)
 \end{aligned}$$

Hence

$$\frac{6A\bar{x}}{L} = -\frac{Wa}{L} (L^2 - a^2)$$

For a series of concentrated loads,

$$\frac{6A\bar{x}}{L} = -\sum \frac{Wa}{L} (L^2 - a^2) \quad \text{Eq. 9.13}$$

where a is the distance of each load in turn from the outer end of the bay.

APPLICATION OF THREE MOMENT EQUATION

For a beam supported at more than three points, such as in Fig. 9.11, we

can derive an equation connecting the moments at any three consecutive supports. In the particular case shown, there will be three such equations, one for ABC, one for BCD and one for CDE. Obviously to solve for all the supports requires two more equations, which will be obtained from the conditions at the outer supports A and E. The possible conditions are as follows:—

(1) *Simple Support*. If the support at A is a simple support at the end of



FIG. 9.11.—BEAM CONTINUOUS OVER MORE THAN THREE SUPPORTS.

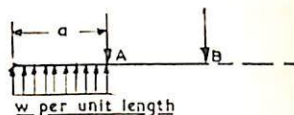


FIG. 9.12.—UNIFORMLY DISTRIBUTED LOAD ON OVERHANGING END OF A BEAM.

the beam, the bending moment there will be zero. Hence, for this type of support we should have the additional equation

$$M_A = 0,$$

with a similar equation for M_E if that were also a simple support at the end of the beam.

(2) *Simple Support, Overhanging End*. In this case M_A will be equal to the moment about A of the load on the overhang. For example, in Fig. 9.12, with a uniformly distributed load, w per unit length, on an overhang of length a ,

$$M_A = \frac{wa^2}{2}, \text{ positive when } w \text{ is positive.}$$

(3) *Fixed Support*. In this case the end moment is an unknown quantity but an additional equation may be obtained by use of an artifice known as the “method of images” or the “method of reflection”. We know that the fixing moment at A (assumed built-in) is equal to the moment produced at A by the forces on the whole of the beam between A and E. Consequently, if we replace the fixation at A by a reflection of the whole beam the equilibrium will be undisturbed (Fig. 9.13).

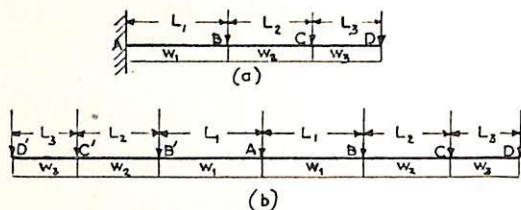


FIG. 9.13.—METHOD OF IMAGES FOR FIXED END SUPPORT.

Furthermore conditions on both sides of A are the same, so that

$$M_{B'} = M_B$$

We can now write another three moment equation for the bay B'AB. Assuming we have already written the equation for ABC, we shall have used $A_1\bar{x}_1$ as the moment of the area of the free moment diagram for AB about A. For B'AB we shall require the moment of this area about B' and B, which will be, in each

case, $A_1(L_1 - \bar{x}_1)$, so that our additional equation is

$$M_B L_1 + 2M_A(2L_1) + M_B L_1 + \frac{6A_1(L_1 - \bar{x}_1)}{L_1} + \frac{6A_1(L_1 - \bar{x}_1)}{L_1} = 0$$

$$\text{i.e.,} \quad M_B L_1 + 2M_A L_1 + \frac{6A_1(L_1 - \bar{x}_1)}{L_1} = 0$$

Application of (1), (2) or (3), whichever is appropriate, to the two outer supports will give us the two additional equations necessary to determine the moments at all the supports.

Having determined the moments at the supports, the resultant bending moment diagram is obtained by algebraic addition of the free and fixing moment diagrams.

The reactions at the supports are obtained as the algebraic sum of the reactions

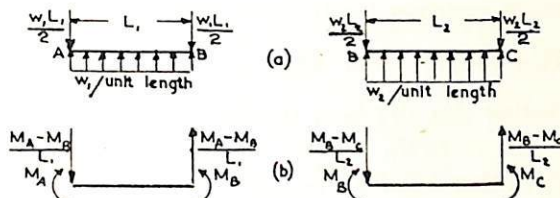


FIG. 9.14.—REACTIONS AT SUPPORTS OF A CONTINUOUS BEAM; (a) "FREE" REACTIONS; (b) FIXING MOMENT REACTIONS.

for free loading and those for the fixing moments. E.g., for uniformly distributed loads, the reactions would be computed as follows (Fig. 9.14):—

$$\left. \begin{aligned} R_A &= \frac{w_1 L_1}{2} + \frac{M_A - M_B}{L_1} \\ R_B &= \frac{w_1 L_1}{2} - \frac{M_A - M_B}{L_1} + \frac{w_2 L_2}{2} + \frac{M_B - M_C}{L_2} \\ &= \frac{w_1 L_1}{2} + \frac{M_B - M_A}{L_1} + \frac{w_2 L_2}{2} + \frac{M_B - M_C}{L_2} \\ R_C &= \frac{w_2 L_2}{2} - \frac{M_B - M_C}{L_2} \\ &= \frac{w_2 L_2}{2} + \frac{M_C - M_B}{L_2} \end{aligned} \right\} \text{Eqs. 9.14}$$

Note that the free reaction and the fixing reactions can be written with a plus sign between them always if the fixing reaction is calculated as

$$(\text{Near moment minus far moment}) \times \frac{1}{L}$$

It may be noted that the reactions are assumed to be in the correct direction—i.e., downwards for a positive load. If we had assumed the reactions to be positive (i.e., upwards), as we usually assume all unknown quantities, then application of the equilibrium conditions would in this case indicate negative reactions (i.e., downwards). As we have written the expressions for the reactions above, a positive value for R indicates a downward reaction.

Numerical examples will illustrate the application of the three moment equation more clearly.

Example 9.2. Fig. 9.15 illustrates a uniform beam with four supports carrying uniformly distributed loads as shown. Determine the moments at the supports B and C, and find the reactions at the four supports (A.F.R.Ae.S.).

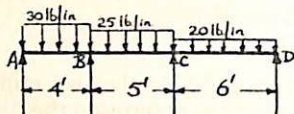


FIG. 9.15.

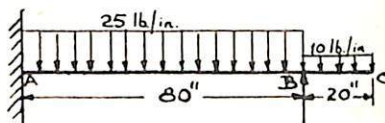


FIG. 9.16.—EXAMPLE 9.3.

The loading in this case is negative, so that the area of the free bending moment diagrams will be positive.

The three moment equation for the part ABC is

$$4M_A + 2M_B(4 + 5) + 5M_C + \frac{360 \times 4^3}{4} + \frac{300 \times 5^3}{4} = 0$$

Simplifying, and noting that $M_A = 0$, this becomes

$$18M_B + 5M_C = -15,135 \quad (1)$$

(It should be observed that w has been expressed in *lb. per ft.*, since L is measured in feet.)

Similarly, for BCD we derive the equation

$$5M_B + 2M_C(5 + 6) + 6M_D + \frac{300 \times 5^3}{4} + \frac{240 \times 6^3}{4} = 0$$

Since $M_D = 0$, this becomes

$$5M_B + 22M_C = -22,335 \quad (2)$$

Solving (1) and (2), we obtain

$$M_B = -596 \text{ lb. ft.}$$

$$M_C = -881 \text{ lb. ft.}$$

Noting that w is negative, the reactions are determined as under.

$$R_A = -\frac{360 \times 4}{2} + \frac{596}{4} = -720 + 149 = -571 \text{ lb.}$$

$$R_B = -720 + \frac{-596}{4} - 750 + \frac{285}{5} = -1562 \text{ lb.}$$

$$R_C = -750 - \frac{285}{5} - 720 - \frac{881}{6} = -1674 \text{ lb.}$$

$$R_D = -720 + \frac{881}{6} = -573 \text{ lb.}$$

The negative signs indicate that the reactions are upwards. As a check, taking moments about D, we have

$$\begin{aligned} M_D &= 571 \times 15 + 1562 \times 11 + 1674 \times 6 - 1440 \times 13 - 1500 \times 8.5 - 1440 \times 3 \\ &= 8565 + 17,182 + 10,044 - 18,720 - 12,750 - 4320 = 0 \end{aligned}$$

Example 9.3. The beam ABC, in Fig. 9.16, is built in at A and supported on a wedge at B. Find the reactions at A and B, and draw the bending moment diagram. (A.F.R.Ae.S.)

This problem could be solved by the method adopted in Example 8.1, but it will be solved here by use of the three moment equation.

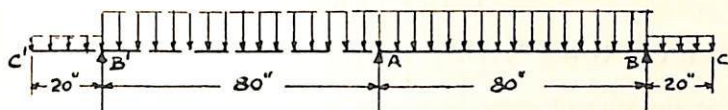


FIG. 9.17.—BEAM OF FIG. 9.16 REFLECTED AT A.

In order to represent the built-in conditions at A, we reflect the beam beyond A as shown in Fig. 9.17. Then, from symmetry we may write,

$$M_{B'} = M_B$$

But
$$M_B = -\frac{10 \times 20^2}{2} = -2000 \text{ lb. in.}$$

Applying the three moment equation to B'AB,

$$80M_{B'} + 320M_A + 80M_B + \frac{25 \times 80^3}{4} + \frac{25 \times 80^3}{4} = 0$$

Hence, inserting the numerical value of $M_{B'}$ and M_B , we obtain

$$4M_A - 4000 + \frac{2 \times 25 \times 80^2}{4} = 0$$

giving

$$M_A = -19,000 \text{ lb. in.}$$

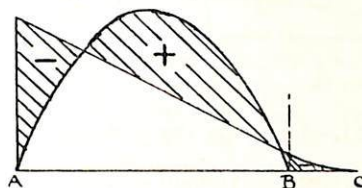


FIG. 9.18.—RESULTANT BENDING MOMENT DIAGRAM FOR BEAM OF FIG. 9.16.

Having determined the fixing moment at A, it is a simple matter to evaluate the reactions at A and B. One way is the following:—

$$R_A = -\frac{25 \times 80}{2} + \frac{M_A - M_B}{80} = -1000 - \frac{17,000}{80} = -1212.5 \text{ lb.}$$

$$R_B = -1000 + \frac{M_B - M_A}{80} - 200 = -987.5 \text{ lb.}$$

The bending moment diagram is as shown in Fig. 9.18.

CHAPTER X

STRUTS—INSTABILITY PROBLEMS

10.1. In the majority of cases in aircraft structures the criterion is not the ultimate strength of the material but the stability of the structure. The problem of instability has long been recognised. In fact, Euler considered the stability of a long slender rod in compression, and published his solution in 1744. The importance of weight-saving in modern aircraft structures has made the consideration of this problem essential.

10.2. The Euler Critical Load for a Pin-Jointed Strut

If a long slender rod is loaded in compression, at a certain value of the end load the rod will bow or buckle. The *critical end load* is sometimes defined as the smallest value of the compressive load which will just maintain the strut in a deflected position. A load smaller than this will not be able to prevent the strut from returning to a straight line, whilst a greater load will cause further deflection and, ultimately, fracture.

Consider a strut ACB deflected as shown in Fig. 10.1. It should be noted that

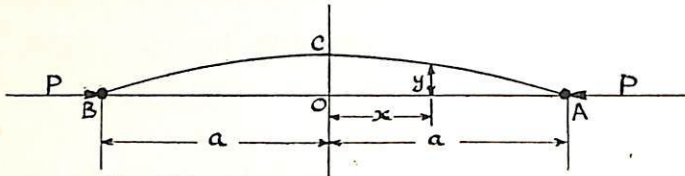


FIG. 10.1.—NOTATION FOR DEFLECTED STRUT.

as usual, we are considering only very small deflections. Taking the origin at O, midway between A and B, measuring x positive to the right from O, we obtain the bending moment due to the end load, P , at x from O as $-Py$, the negative sign following the convention adopted in Volume I. Hence if Young's Modulus for the material is E and the second moment of area of the cross-section of the strut (assumed uniform) is I , we may write

$$EI \frac{d^2 y}{dx^2} = -Py$$

Writing $\mu^2 = P/EI$ this becomes

$$\frac{d^2 y}{dx^2} + \mu^2 y = 0 \quad \dots \quad \text{Eq. 10.1}$$

The solution to this equation, as may be verified by substitution, is

$$y = A \cos \mu x + B \sin \mu x \quad \dots \quad \text{Eq. 10.2}$$

where A and B are constants to be determined.

We know that $y = 0$ at the two ends of the strut, where $x = \pm a$. Hence for $x = a$,

$$0 = A \cos \mu a + B \sin \mu a$$

and for $x = -a$,

$$0 = A \cos \mu a - B \sin \mu a$$

These two equations lead to

$$A \cos \mu a = 0$$

$$B \sin \mu a = 0$$

These equations must be satisfied simultaneously, so let us investigate the conditions under which this requirement may be met.

(i) $A = 0$ and $B = 0$. This solution would not meet our case as it means simply that the deflection is zero everywhere.

(ii) $A = 0$ and $\sin \mu a = 0$. This gives $\mu a = 0$ or π or $2\pi \dots$ and hence $P = 0$ or $\frac{\pi^2 EI}{a^2} \dots$

(iii) $B = 0$ and $\cos \mu a = 0$. This gives $\mu a = \pi/2$ or $3\pi/2 \dots$ and hence $P = \pi^2 EI/4a^2$ or $9\pi^2 EI/4a^2 \dots$

Consideration of (i), (ii) and (iii) shows that the smallest finite value of P satisfying the conditions of the problem is

$$P_e = \pi^2 EI/4a^2 = \pi^2 EI/L^2 \dots \text{Eq. 10.3}$$

where $L = 2a$ = length of strut. This load is the *Euler critical load* for a pin-jointed strut. Theoretically the higher values of P obtained above correspond to more complex forms of the deflected strut, which would in practice not be reached

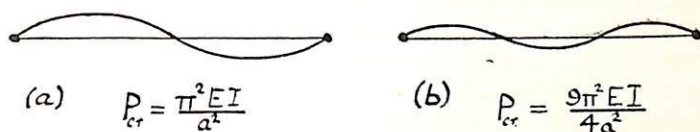


FIG. 10.2.—MODES OF DEFLECTION OF STRUT.

as the material would almost certainly fail first. The forms corresponding to two higher values of the critical load P_{cr} are shown in Fig. 10.2. The minimum value, denoted by P_e , will be understood when we refer in future to the Euler load.

10.3. The Euler Critical Stress for a Pin-Jointed Strut

If the cross-sectional area of the strut is A , the compressive stress, p , due to a compressive load P is given by

$$p = P/A$$

In particular the stress, p_e , corresponding to the Euler load, P_e , will be

$$p_e = P_e/A = \pi^2 EI/L^2 A$$

Noting that $I/A = k^2$ where k is the radius of gyration of the cross-section, this expression may be written

$$p_e = \pi^2 EI/(L/k)^2 \dots \text{Eq. 10.4}$$

The quantity (L/k) is termed the *slenderness ratio* of the strut, and is a convenient parameter against which we can plot p_e . It should be emphasised that the value of k to be used is the smallest value of the radius of gyration for the cross-section. This takes account of the fact that the strut will bend about the neutral axis for which I has its least value.

For struts of low slenderness ratio it is obvious that p_e will be very large, and, as a first approximation, the limiting practical value of the critical stress will be given by the compressive failing stress of the material—usually taken as the 0.2% proof stress. A curve of critical stress plotted against slenderness ratio for a

pin-jointed strut, assuming failure to be due either to Euler type buckling or to failure of the material in compression would then be of the form shown in Fig. 10.3. In practice it is found that imperfections in manufacture, slight lack of straightness, and inability to guarantee a truly axial load combine to prevent realisation of the ideal Euler critical load except with long slender struts. Above a value of L/k of about 130 the Euler curve may be used with a fair degree of accuracy, and below

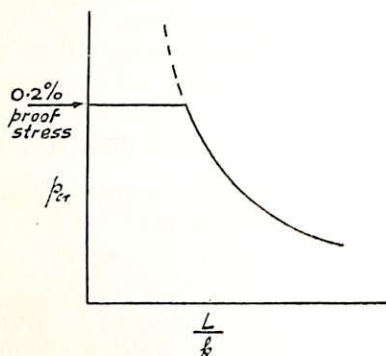


FIG. 10.3.—FORM OF CURVE OF CRITICAL STRESS PLOTTED AGAINST SLENDERNESS RATIO.

that figure empirical curves are generally used. Before discussing briefly some of these empirical curves, the effect on the Euler critical load of types of end attachment other than pin joints will be considered, as will the effect of eccentricity of end load and the effect of initial bow of the strut.

10.4. The Euler Strut with Various End Attachments

(a) BOTH ENDS FIXED

Fixation of the ends of the strut introduces a fixing moment M_0 at each end,

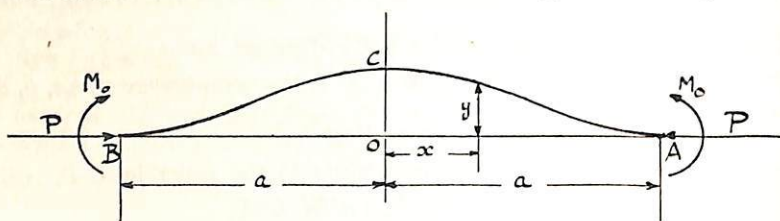


FIG. 10.4.—STRUT WITH BOTH ENDS FIXED.

with the condition that the slope, dy/dx , is zero at each end. With the notation of Fig. 10.4, the basic equation for bending is now

$$EI \frac{d^2 y}{dx^2} = -Py + M_0$$

and the solution is

$$y = A \cos \mu x + B \sin \mu x + M_0/\mu^2 EI$$

where $\mu^2 = P/EI$.

From this we have

$$\frac{dy}{dx} = -\mu A \sin \mu x + \mu B \cos \mu x$$

Applying the condition that dy/dx is zero when $x = \pm a$ we obtain

$$0 = -\mu A \sin \mu a + \mu B \cos \mu a$$

and

$$0 = +\mu A \sin \mu a + \mu B \cos \mu a$$

These two equations result in

$$B = 0 \quad \text{and} \quad \mu A \sin \mu a = 0$$

The smallest finite value of μa satisfying the conditions is therefore the smallest value of μa such that $\sin \mu a = 0$. Hence the critical end load is given by

$$\mu a = \pi$$

so that

$$P_{cr} = \pi^2 EI / a^2 = 4\pi^2 EI / L^2 = 4P_e$$

Hence the effect of applying complete fixation to the ends of the strut is to make the critical load four times the Euler load for a pin-jointed strut. This could be expressed also by saying that the effect is to reduce the effective length to $L/2$ for use in the Euler formula for the critical load.

(b) ONE END FIXED, THE OTHER FREE

Assuming the end B fixed and the end A free, a possible deflected form of the

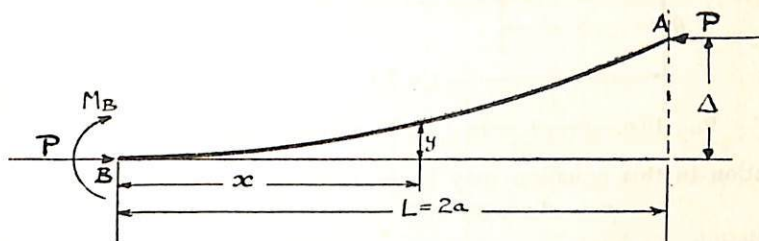


FIG. 10.5.—STRUT WITH ONE END FIXED, THE OTHER FREE.

strut is shown in Fig. 10.5. If we measure x and y as shown, the deflection at A being Δ , we may write

$$EI \frac{d^2 y}{dx^2} = P(\Delta - y)$$

i.e.,
$$\frac{d^2 y}{dx^2} + \mu^2 y = \mu^2 \Delta$$

The solution to this equation is

$$y = A \cos \mu x + B \sin \mu x + \Delta$$

By differentiation we obtain

$$\frac{dy}{dx} = -\mu A \sin \mu x + \mu B \cos \mu x$$

Applying the conditions that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$ we see that $A = -\Delta$ and $B = 0$

Hence

$$y = \Delta(1 - \cos \mu x)$$

But at A, where $x = 2a = L$, we know that $y = \Delta$, so that

$$\Delta = \Delta(1 - \cos \mu L)$$

i.e.,

$$\cos \mu L = 0$$

and

$$\mu L = \pi/2$$

giving

$$P_{cr} = \pi^2 EI / 4L^2 = P_e / 4$$

In this case, therefore, the critical load is one quarter of the Euler load for a pin-jointed strut of the same length. Alternatively, the effective length is twice that of the pin-jointed strut.

(c) ONE END FIXED, THE OTHER PINNED

Measuring x and y as indicated in Fig. 10.6, taking the pinned end B as origin, the basic equation is

$$EI \frac{d^2 y}{dx^2} = Rx - Py$$

or

$$\frac{d^2 y}{dx^2} + \mu^2 y = \frac{Rx}{EI}$$

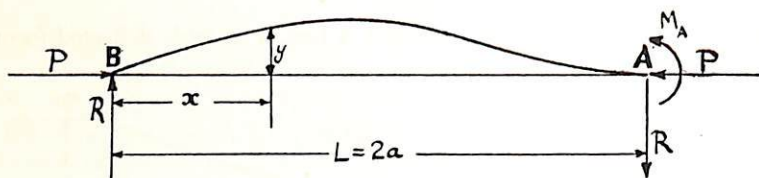


FIG. 10.6.—STRUT WITH ONE END FIXED, THE OTHER PINNED.

The solution to this equation may be written

$$y = A \cos \mu x + B \sin \mu x + Rx / \mu^2 EI$$

Differentiating, we have

$$\frac{dy}{dx} = -\mu A \sin \mu x + \mu B \cos \mu x + R / \mu^2 EI$$

We know that when $x = 0$, $y = 0$ and it follows that $A = 0$. Further, when $x = L$, both $y = 0$ and $\frac{dy}{dx} = 0$ so that

$$0 = B \sin \mu L + RL / \mu^2 EI$$

and

$$0 = \mu B \cos \mu L + R / \mu^2 EI$$

From these two equations we obtain

$$\tan \mu L = \mu L$$

This equation may be solved graphically, and we find that the smallest finite value of μL satisfying the conditions of the problem is

$$\mu L = 4.493 \text{ radians}$$

Hence the critical value of the end load is given by

$$P_{cr} = \frac{(4.493)^2 EI}{L^2} \simeq \frac{2\pi^2 EI}{L^2}$$

so that the critical end load is approximately twice the Euler load for a pin-jointed strut of the same length, and the effective length is approximately $L/\sqrt{2}$ or $0.707L$.

The results obtained above for the various types of end attachments may be summarised as in Fig. 10.7.

In practice, fixed ends will be difficult to obtain in aircraft structures, and it is often decided to use the Euler load for a pin-jointed strut even when it is obvious that there will be some fixation of the ends. The degree of fixation of the ends may be represented by a "fixity coefficient," c , defined by the equation

$$P_c = \frac{c\pi^2 EI}{L^2} \quad \dots \quad \text{Eq. 10.5}$$

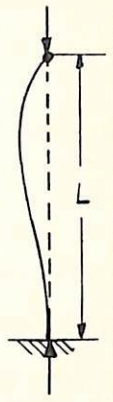
END ATTACHMENT	BOTH ENDS PINNED	BOTH ENDS FIXED	ONE END FIXED ONE END FREE	ONE END FIXED ONE END PINNED
DEFLECTION CURVE				
EULER LOAD P_c	$\frac{\pi^2 EI}{L^2}$	$\frac{4\pi^2 EI}{L^2}$	$\frac{\pi^2 EI}{4L^2}$	$\frac{2\pi^2 EI}{L^2}$ approx.
EFFECTIVE LENGTH	L	$\frac{L}{2}$	$2L$	$0.707L$ approx

FIG. 10.7.—STRUTS WITH VARIOUS END ATTACHMENTS.

The value of c varies from 1 for pin-jointed struts to 4 for struts with fully fixed ends. In aircraft structures a value of c greater than 2 would not normally be used. It is generally considered that the degree of fixation is so uncertain that it is preferable to assume pinned ends, with c equal to unity, even at the expense of some extra weight. Fig. 10.8 shows graphically the effect of different end attachments on the Euler failing stress for struts of material for which $E = 10 \times 10^6$ lb. per sq. in. and the 0.2% proof stress = 47,000 lb. per sq. in.

As mentioned previously the Euler formula gives too high a value of the failing

load for a strut in practice except for very long slender struts. It is not generally possible to guarantee either truly axial loading or complete straightness of the

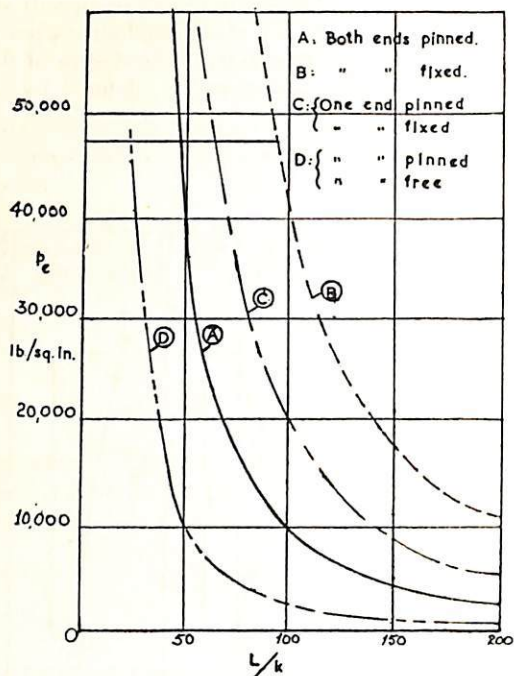


FIG. 10.8.—EFFECT OF END ATTACHMENTS ON EULER STRESS. THE CURVES ARE DRAWN FOR $E = 10 \times 10^6$ LB. PER SQ. IN. 0.2% PROOF STRESS = 47,000 LB. PER SQ. IN.

strut. In the next two sections we shall investigate the effects of eccentricity of the end load and of initial curvature of the strut.

10.5. Strut with Eccentrically Applied End Load

The end load P is assumed to be applied in a line offset from the axis of the strut by an amount e (see Fig. 10.9). Assuming, as before, that the strut is held in a deflected position by the load P , we have

$$EI \frac{d^2 y}{dx^2} = -P(e + y)$$

or
$$\frac{d^2 y}{dx^2} + \mu^2 y = -\mu^2 e$$

The solution is
$$y = A \cos \mu x + B \sin \mu x - e$$

We know that $y = 0$ when $x = \pm a$ so that

$$0 = A \cos \mu a + B \sin \mu a - e$$

and

$$0 = A \cos \mu a - B \sin \mu a - e$$

It follows from these two equations that

$$B = 0 \quad \text{and} \quad A = e \sec \mu a$$

Then

$$y = e(\sec \mu a \cos \mu x - 1)$$

In particular the deflection at the centre of the strut, where $x = 0$, is

$$y_{\max.} = e(\sec \mu a - 1)$$

so that the maximum bending moment is given by

$$M_{\max.} = -P(e + y_{\max.}) = -Pe \sec \mu a$$

Consequently, if I is the second moment of area of the strut cross-section about the neutral axis about which bending is assumed to take place, and if h is the

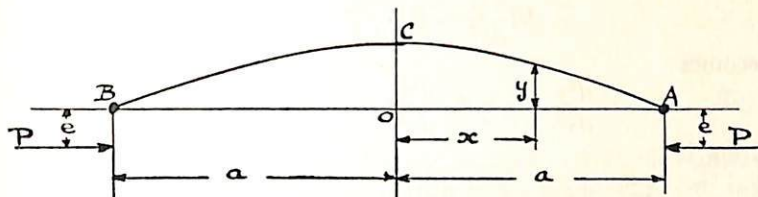


FIG. 10.9.—ECCENTRICALLY LOADED STRUT.

distance from the neutral axis to the extreme fibre of the section, whilst A is the area of the cross-section, the maximum compressive stress will be given by

$$f_{\max.} = \frac{P}{A} + \frac{(Pe \sec \mu a)h}{I}$$

Writing $I = Ak^2$, where k is the radius of gyration of the cross-section,

$$f_{\max.} = \frac{P}{A} \left[1 + \frac{he \sec \mu a}{k^2} \right] \quad \text{Eq. 10.6}$$

If now the maximum compressive stress is not to exceed a certain value, denoted by p_c , we have

$$P = \frac{Ap_c}{1 + \frac{he \sec \mu a}{k^2}} \quad \text{Eq. 10.7}$$

as an expression giving the allowable end load. This equation cannot be solved directly, P being contained implicitly on the right-hand side of the equation in μa , and a graphical solution is usually adopted. It is customary to present information regarding critical loads or stresses by means of strut curves, which often take the form of graphical solutions of the equation just derived, in which a value of e based largely on experiment is used.

10.6. Strut with Initial Curvature

It is necessary to assume a form of initial deflection for the strut, and, for small

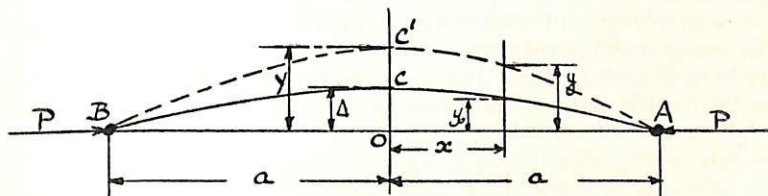


FIG. 10.10.—STRUT WITH INITIAL CURVATURE.

deflections, it will generally be sufficiently accurate to take the deflected shape as being given by

$$y_0 = \Delta \cos \frac{\pi x}{L} = \Delta \cos \frac{\pi x}{2a}$$

Δ is then the initial deflection at the mid-point (Fig. 10.10). In this case the bending moment at any point will be proportional to the *change* in curvature, so that

$$EI \frac{d^2 y}{dx^2} - EI \frac{d^2 y_0}{dx^2} = -Py$$

which becomes

$$\frac{d^2 y}{dx^2} + \mu^2 y = \frac{d^2 y_0}{dx^2} = -\frac{\pi^2 \Delta}{4a^2} \cos \frac{\pi x}{2a}$$

The solution is

$$y = A \cos \mu x + B \sin \mu x + \frac{\pi^2 \Delta}{\pi^2 - 4a^2 \mu^2} \cos \frac{\pi x}{2a}$$

Applying the condition that $y = 0$ when $x = \pm a$, we see that $A = B = 0$ so that

$$y = \frac{\pi^2 \Delta}{\pi^2 - 4a^2 \mu^2} \cos \frac{\pi x}{2a}$$

The deflection at the centre, Y , is given by

$$\begin{aligned} Y &= \frac{\pi^2 \Delta}{\pi^2 - 4a^2 \mu^2} \\ &= \Delta \cdot \frac{\pi^2}{\pi^2 - 4a^2 \frac{P}{EI}} = \Delta \cdot \frac{P_e}{P_e - P} \quad \dots \quad \text{Eq. 10.8} \end{aligned}$$

The maximum bending moment is $M_0 = -PY$, and the maximum compressive stress is given by

$$f_{\max.} = \frac{P}{A} + \frac{M_0 h}{I} \quad \dots \quad \text{Eq. 10.9}$$

It is apparent that initial curvature of the strut has an effect similar to that of eccentricity of the end load. Consequently a combination of an eccentric end load and a strut with initial curvature may be represented by an "equivalent eccentricity," and this is done in some empirical strut formulæ.

10.7. Experimental Determination of the Euler Failing Load

Since it is not possible to guarantee perfect straightness of a strut, usually a strut on test will fail at a load less than the theoretical Euler critical load, and direct determination of P_e by experiment is rarely possible. By adopting a method of analysis and plotting of experimental results suggested by R. V. Southwell, it is, however, possible to determine P_e indirectly.

It has been shown in para. 10.6 that, to a sufficient degree of accuracy, the deflection at the mid-point of the strut in the neighbourhood of the failing load is given by

$$Y = \frac{\Delta \cdot P_e}{P_e - P} = \Delta \cdot \frac{1}{1 - \frac{P}{P_e}}$$

Then δ , the deflection of the mid-point of the strut from its initial position, is given by

$$\begin{aligned}\delta &= Y - \Delta = \Delta \left[\frac{1}{1 - \frac{P}{P_e}} - 1 \right] \\ &= \Delta \cdot \frac{\frac{P}{P_e}}{\frac{P}{P_e} - 1}\end{aligned}$$

so that

$$\delta = P_e \cdot \frac{\delta}{P} - \Delta \quad \dots \quad \text{Eq. 10.10}$$

It follows from this equation that if δ is plotted against δ/P , the resulting graph, in the neighbourhood of the failing load, will be a straight line with slope equal

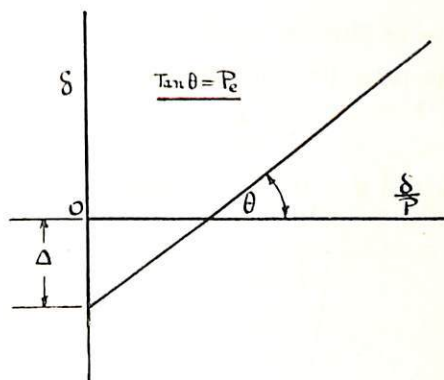


FIG. 10.11.—PLOT OF EQUATION 10.10.

to P_e and with an intercept on the δ axis of $-\Delta$. It must be emphasised that this straight line will result only from a plot of values near the failing load, but it will generally be found that sufficient points can be determined to define the straight line portion. Fig. 10.11 shows the graphical representation of equation 10.10.

10.8. Practical Applications

(i) *Use of the Tangent Modulus.* In the foregoing analysis it has been assumed that the stress-strain curve for the material is a straight line up to the proof stress. In general, of course, this will not be the case, and the expression for the Euler critical load should be modified by substitution of the tangent modulus, E_T , for Young's Modulus, E . The tangent modulus is the slope of the stress-strain curve, and within the elastic limits is equal to E . The modification to the curve of Euler stress, p_e , plotted against slenderness ratio is indicated in Fig. 10.12.

(ii) *Arbitrary Expressions.* There are many arbitrary expressions in use, most of which have been derived for use in civil engineering structures and which are not generally suitable for aircraft structures. One means of assessing the failing load for struts of hollow circular section is to use an equivalent eccentricity in the

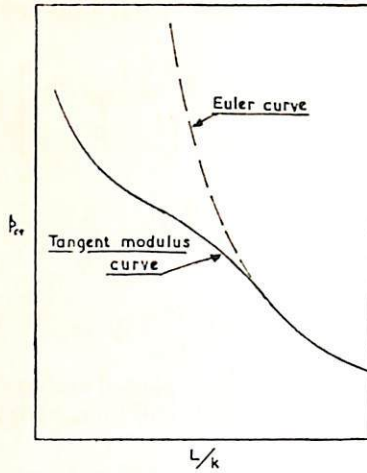


FIG. 10.12.—USE OF TANGENT MODULUS IN DRAWING STRUT CURVE.

expression developed in para. 10.5, and one suggested value for this equivalent eccentricity is as follows :—

$$e = \frac{L}{600} + \frac{d}{40}$$

where L is the length, and d is the inside diameter.

Another method sometimes adopted for steel tubes in aircraft structures is to

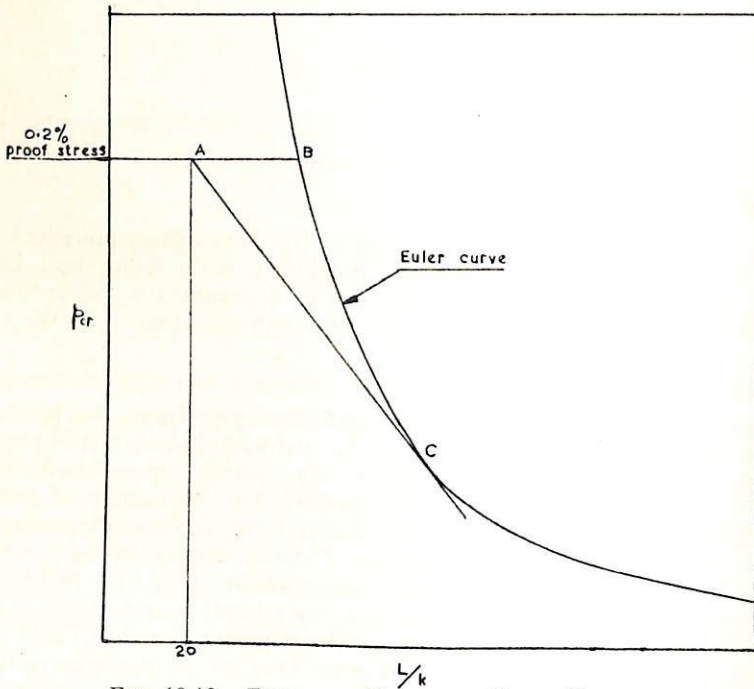


FIG. 10.13.—EMPIRICAL CURVE FOR STEEL TUBES.

draw the line AC tangential to the Euler curve, starting from the point A where L/k is 20 and p is the proof stress, as shown in Fig. 10.13. This line is then used for values of L/k below that at the point C.

All the various methods adopted for predicting the failing loads of columns or struts are attempts at defining the strut curve in the short strut region—i.e., for small values of L/k . There is a fair amount of experimental evidence to suggest that the use of the tangent modulus in the Euler curve will give results of reasonable accuracy in all cases where the stress-strain curve departs from a straight line at a stress well below the proof stress.

10.9. Other Types of Instability

(i) *Torsional Instability.* The theory of this type of instability is too complicated to be included here, but it is important that its existence should be recognised, since it is a characteristic of open sections such as are used to a large extent in aircraft structures. The type of failure is depicted in Fig. 10.14, which shows a

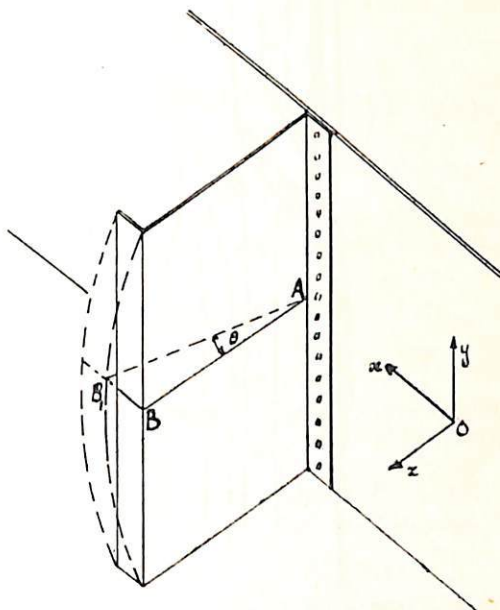


FIG. 10.14.—TORSIONAL INSTABILITY.

Z-section stringer or stiffener attached along one flange to a member which is sufficiently rigid to prevent the stiffener from bowing about the axis Ox . The free flange, when the stiffener is subjected to a compressive end load, will bend about the axis Oz , deflecting in the direction Ox . A cross-section AB will move to the position AB_1 , the effect being that of a rotation of the cross-section through an angle θ , as shown in Fig. 10.15. It is this effect which gives

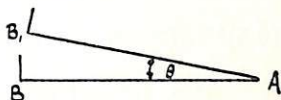


FIG. 10.15.—ROTATION OF CROSS-SECTION OF STRUT IN FIG. 10.14.

rise to the name *torsional instability*. Fig. 10.16 is a drawing of an actual test specimen which failed in this manner under compression.

(ii) *Local Instability*. Many of the struts used in aircraft structures are of thin sheet construction, and failure may occur by local buckling or wrinkling of the thin sheet as opposed to instability of the strut as a whole. This type of failure, usually termed secondary failure, is independent of the length of the strut, and will be

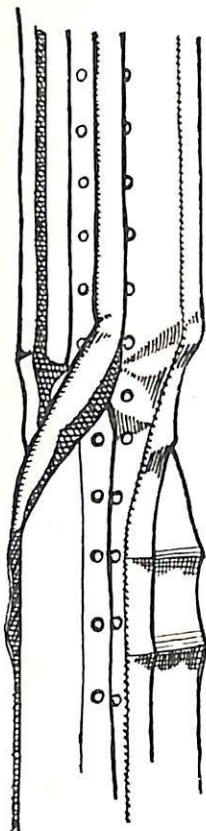


FIG. 10.16.—EXAMPLE OF TORSIONAL INSTABILITY FAILURE.

mentioned in a little more detail in a later chapter. It suffices to point out here that the local instability stress of a thin sheet is a function of the thickness/width ratio, and depends also on the type of edge support. Some formulæ for the critical stress will be given later.

Example 10.1. Calculate the Euler failing load for a strut, which is in the form of a hollow tube, 20 in. long, 0.5 in. diameter, with walls 0.1 in. thick. Take $E = 10^7$ lb. per sq. in.

$$I = \frac{\pi}{64}[(0.5)^4 - (0.3)^4] = 0.00267 \text{ in.}^4$$

$$P_e = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 10^7 \times 0.00267}{20^2} = 659 \text{ lb.}$$

Example 10.2. A strut has the following mechanical properties:—

Length between pin-jointed ends	30.0 in.
Radius of gyration	1.0 in.
Cross-sectional area	0.3 sq. in.
Distance of outermost fibres from neutral surface	1.2 in.
Young's modulus	10^7 lb. per sq. in.

It is found by test that the strut fails under an end load of 6,000 lb. when this load is applied to the strut eccentrically, the eccentricity of loading being 0.6 in. Estimate the load which would cause failure if the eccentricity were increased to 1.2 in. (A.F.R.Ae.S.)

Application of the test results to equation 10.6 will give the maximum compressive stress for the material, and it will then be possible to estimate the load which will cause this stress to be reached with an eccentricity of 1.2 in.

For the test results

$$\mu a = \left(\frac{6000}{10^7 \times 0.3 \times 1.0} \right)^{1/2} \times 15 = 0.67 \text{ radian}$$

$$\sec \mu a = 1.275$$

$$f_{\max.} = \frac{6000}{0.3} \left(1 + \frac{0.6 \times 1.2 \times 1.275}{1.0} \right) = 38,400 \text{ lb. per sq. in.}$$

Substituting this value in equation 10.7 for p_c and using the increased eccentricity,

$$P = \frac{A p_c}{1 + \frac{e h \sec \mu a}{k^2}} = \frac{11,520}{1 + 1.44 \sec (0.00866 \sqrt{P})}$$

This equation may be written in the form

$$\sec (0.00866 \sqrt{P}) = \frac{7990}{P} - 0.694$$

This equation may be solved for P either graphically, by plotting the expressions on the two sides separately against P , or by a method of continuous approximation. We shall adopt the latter method here.

As a first approximation, we guess at a value of P . Let us assume $P = 4000$ lb. Putting this value in the left-hand side of the equation only, the equation becomes

$$1.172 = \frac{7990}{P} - 0.694$$

and solution of this gives $P = 4280$ lb. as our second approximation. Inserting this new value in the left-hand side of the equation gives

$$1.185 = \frac{7990}{P} - 0.694$$

the solution of which is $P = 4250$ lb.

It is obvious that the approximations are converging rapidly, and the value of 4250 lb. will be taken as the solution.

STRUTS WITH LATERAL LOADING

11.1. General

STRUTS which carry lateral loads as well as axial compressive loads—or, alternatively, beams subjected to axial compression, sometimes termed *beam columns*—occur frequently in aircraft structures, and it is necessary to take account of the additional bending moments arising in such cases. A beam carrying a lateral load will deflect under that load. If now a compressive load P is applied, as indicated in Fig. 11.1, an additional bending moment of magnitude Py , where y is the deflection, will

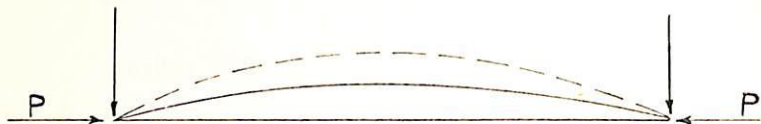


FIG. 11.1.—ADDITIONAL DEFLECTION OF A BEAM DUE TO AXIAL LOAD, P .

be introduced at each point of the beam. This *secondary bending moment* will produce further deflection and hence further bending moment, and this process will continue until either an equilibrium position is reached or the beam fractures. Provided P is less than a certain critical value, equilibrium will be reached if the material is strong enough to withstand the imposed bending moments. It can be shown that the critical value of P is the Euler load considered in Chapter X. If P is less than this critical value, the problem becomes one of determining the bending moments in the beam, making due allowance for the deflection. For simple types of loading, a graphical method known as the “polar bending moment diagram,” due to H. B. Howard, is perhaps the most convenient, but, before describing this method, the analytical solution will be discussed briefly. In view of the complexity of the formulæ resulting from an analytical treatment of the problem, several approximate methods have been suggested, but few of these are of great value for the design of aircraft structures because of the necessity for keeping the structure weight down to a minimum.

11.2. Analytical Solution

We consider the beam AB, in Fig. 11.2, which carries a distributed lateral load, and an axial compressive load P , together with end moments M_A and M_B . If y is the deflection at distance x from the centre, O, of the beam, we may write the

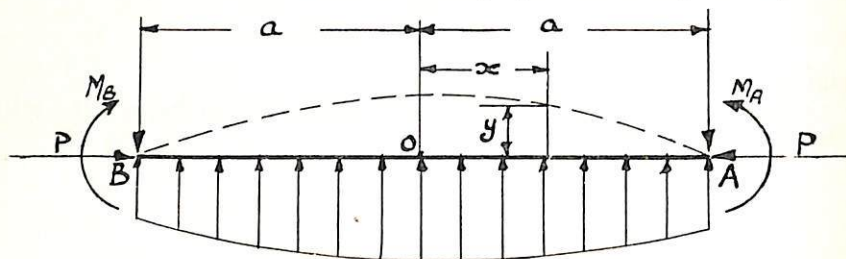


FIG. 11.2.—STRUT WITH LATERAL LOADING—NOTATION.

bending moment at this point as

$$M = M_1 - Py \quad \text{Eq. 11.1}$$

where M_1 is the bending moment due to the lateral loading and the end moments alone.

Differentiating equation 11.1 with respect to x , we obtain

$$\frac{dM}{dx} = \frac{dM_1}{dx} - P \frac{dy}{dx} \quad \text{Eq. 11.2}$$

We note that $\frac{dM}{dx}$ = shear force = S

and $\frac{dM_1}{dx}$ = shear corresponding to the *primary bending moment*, M_1
 $= S_1$

so that equation 11.2 may be written

$$S = S_1 - Pi \quad \text{Eq. 11.3}$$

where $i = \frac{dy}{dx}$ = the slope of the beam at the point considered. M_1 and S_1 are often termed the "apparent bending moment" and the "apparent shear" respectively.

Differentiating equation 11.2,

$$\frac{d^2M}{dx^2} = \frac{d^2M_1}{dx^2} - P \frac{d^2y}{dx^2}$$

Remembering that $M = EI \frac{d^2y}{dx^2}$, this becomes

$$\frac{d^2M}{dx^2} = \frac{d^2M_1}{dx^2} - \frac{PM}{EI}$$

and, writing $\mu^2 = \frac{P}{EI}$, we have finally,

$$\frac{d^2M}{dx^2} + \mu^2 M = \frac{d^2M_1}{dx^2} \quad \text{Eq. 11.4}$$

This equation is of the same form as those obtained in Chapter X with y replaced by M . If M_1 can be expressed conveniently as a function of x the term $\frac{d^2M_1}{dx^2}$ may be obtained also as a function of x . In the general case, let $\frac{d^2M_1}{dx^2} = f(x)$. Particular cases will be considered later. Equation 11.4 now becomes

$$\frac{d^2M}{dx^2} + \mu^2 M = f(x)$$

This differential equation may be solved by use of the D -operator to give

$$M = A \cos \mu x + B \sin \mu x + \frac{1}{D^2 + \mu^2} f(x) \quad \text{Eq. 11.5}$$

where $D^2 = \frac{d^2}{dx^2}$, and A and B are constants.

The last term in equation 11.5 may be evaluated in the following manner.

$$\frac{1}{D^2 + \mu^2} = \frac{1}{\mu^2} \left[1 + \frac{D^2}{\mu^2} \right]^{-1} = \frac{1}{\mu^2} \left(1 - \frac{D^2}{\mu^2} + \frac{D^4}{\mu^4} - \frac{D^6}{\mu^6} + \dots \right)$$

so that

$$g(x) = \frac{1}{D^2 + \mu^2} f(x) = \frac{1}{\mu^2} \left[f(x) - \frac{1}{\mu^2} \frac{d^2 f(x)}{dx^2} + \frac{1}{\mu^4} \frac{d^4 f(x)}{dx^4} - \dots \right] \quad \text{Eq. 11.6}$$

Consequently we may write our solution as

$$M = A \cos \mu x + B \sin \mu x + g(x) \quad \text{Eq. 11.7}$$

The constants A and B may be determined from the boundary conditions. We have that $M = M_A$ when $x = a$, and $M = M_B$ when $x = -a$, and substitution of these values into equation 11.7 gives the two equations

$$M_A = A \cos \mu a + B \sin \mu a + g(a) \quad \text{Eq. 11.8}$$

$$M_B = A \cos \mu a - B \sin \mu a + g(-a) \quad \text{Eq. 11.9}$$

If the values of A and B are determined from equations 11.8 and 11.9, and substituted into equation 11.7, we have an equation giving the bending moment at any point of the beam, and the maximum compressive stress at any point may then be determined as indicated in para. 10.5.

For particular types of loading it is necessary to evaluate the functions $f(x)$ and $g(x)$, and we proceed to do this.

(i) END MOMENTS ONLY, NO LATERAL LOADING (Fig. 11.3)

The reactions at the supports due to the end moments will be $\pm \frac{M_A - M_B}{2a}$

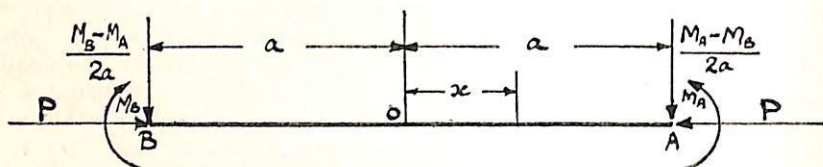


FIG. 11.3.—STRUT WITH END MOMENTS.

as shown in the figure. The apparent bending moment at x from the centre, O , is therefore given by

$$M_1 = M_A - \frac{M_A - M_B}{2a}(a - x)$$

Hence,
$$f(x) = \frac{d^2 M_1}{dx^2} = 0$$

in this case, and it follows that

$$g(x) = 0 \text{ also.}$$

Equations 11.8 and 11.9 become

$$M_A = A \cos \mu a + B \sin \mu a$$

$$M_B = A \cos \mu a - B \sin \mu a$$

and we obtain

$$A = \frac{M_A + M_B}{2 \cos \mu a} \quad \text{and} \quad B = \frac{M_A - M_B}{2 \sin \mu a}$$

Substituting into equation 11.7 gives

$$M = \left(\frac{M_A + M_B}{2 \cos \mu a} \right) \cos \mu x + \left(\frac{M_A - M_B}{2 \sin \mu a} \right) \sin \mu x \quad \text{Eq. 11.10}$$

(ii) END MOMENTS AND UNIFORMLY DISTRIBUTED LATERAL LOAD

For a uniformly distributed lateral load of w per unit length, and end moments

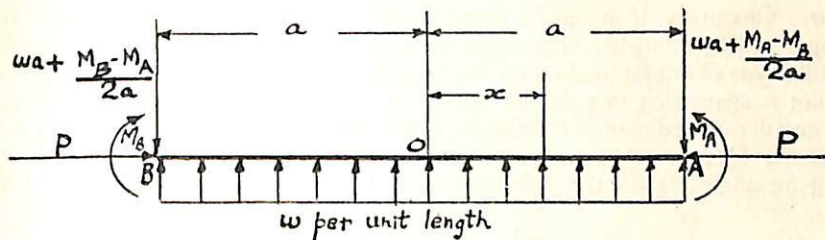


FIG. 11.4.—STRUT WITH END MOMENTS AND UNIFORMLY DISTRIBUTED LATERAL LOAD.

M_A and M_B the reactions at the supports are as indicated in Fig. 11.4, and at x from the centre of the beam

$$M_1 = M_A + \frac{w(a-x)^2}{2} - \left(wa + \frac{M_A - M_B}{2a} \right) \cdot x$$

Then

$$f(x) = \frac{d^2 M_1}{dx^2} = w$$

and

$$g(x) = \frac{w}{\mu^2}$$

For this case, equations 11.8 and 11.9 become

$$M_A = A \cos \mu a + B \sin \mu a + w/\mu^2$$

$$M_B = A \cos \mu a - B \sin \mu a + w/\mu^2$$

Hence

$$A = \frac{M_A + M_B - 2w/\mu^2}{2 \cos \mu a}$$

and

$$B = \frac{M_A - M_B}{2 \sin \mu a}$$

Then

$$M = \frac{M_A + M_B - 2w/\mu^2}{2 \cos \mu a} \cos \mu x + \frac{M_A - M_B}{2 \sin \mu a} \sin \mu x + \frac{w}{\mu^2} \quad \text{Eq. 11.11}$$

It will be readily observed that a purely analytical treatment leads to rather complicated expressions, and in other cases, such as that in which the lateral loading is a concentrated load applied at a point offset from the centre of the beam, the problem becomes more involved because of the discontinuity in the bending moment which arises. It is not proposed to proceed further with the analytical

treatment, the general procedure having been indicated in the foregoing discussion. A more convenient method of solution in most cases is the Howard polar diagram, which will be considered in the following paragraphs.

11.3. Howard Polar Diagrams

Equation 11.7 could have been written in the form

$$M = C \cos(\mu x - \epsilon) + g(x) \quad \text{Eq. 11.12}$$

where C and ϵ are constants determined by the boundary conditions. If we write

$$m = M - g(x) \quad \text{Eq. 11.13}$$

then m has the dimensions of a bending moment, and we call it the *difference moment*. Obviously, if m can be determined graphically, or by any other means, the true bending moment, M , may be evaluated, since $g(x)$ will be known for the particular type of lateral loading on the beam. The method suggested by Howard is to plot m against μx in polar co-ordinates, using m as the radius vector and μx as the angular co-ordinate. Plotting equation 11.13 in this way, we obtain a circle, of diameter C , passing through the origin, with the diameter through the origin making an angle ϵ with the reference line, OZ , as illustrated in Fig. 11.5. In

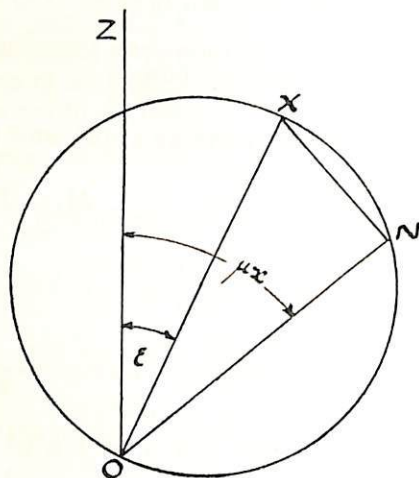


FIG. 11.5.—BASIS OF HOWARD POLAR DIAGRAM. PLOT OF EQUATION 11.13.

this figure, $OX = C$, the angle $ZOX = \epsilon$, and N is any point on the circle on OX as diameter such that the angle $ZON = \mu x$. Then $\angle X'NO$ is a right-angle, and $\angle X'ON = \mu x - \epsilon$. The point X is called the *apex* of the diagram.

We see that

$$\begin{aligned} ON &= OX \cos \angle X'ON \\ &= C \cos(\mu x - \epsilon) \end{aligned}$$

i.e., the radius vector ON gives the value of m satisfying equation 11.13. If we differentiate equation 11.13, we obtain

$$\frac{dm}{dx} = \frac{dM}{dx} - \frac{d}{dx}g(x) = -\mu C \sin(\mu x - \epsilon)$$

Since $\frac{dM}{dx}$ = shear force, we call $\frac{dm}{dx}$ the *difference shear*, and denote it by s , so that

$$s = S - \frac{d}{dx}g(x) = -\mu C \sin(\mu x - \epsilon) = -\mu \cdot XN. \quad \text{Eq. 11.14}$$

Hence the shear force may also be determined from Fig. 11.5.

The polar diagram for a beam of length $2a$ will be as shown in Fig. 11.6, the

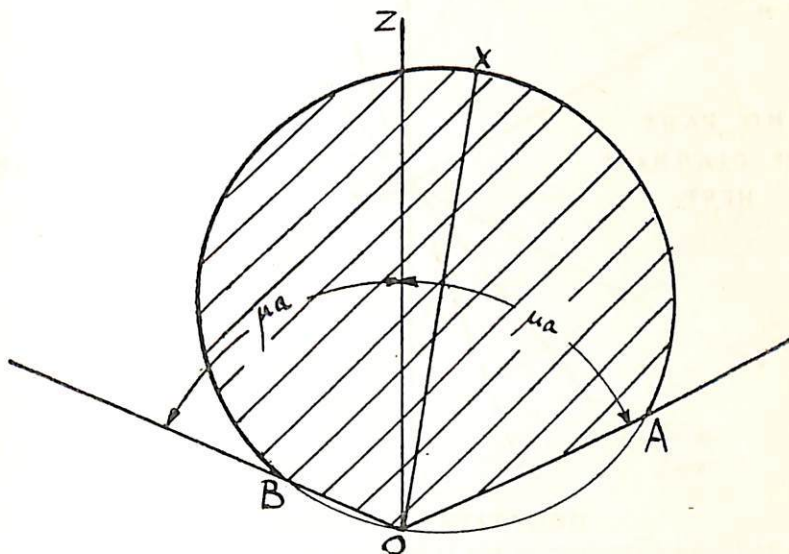


FIG. 11.6.—BASIC POLAR DIAGRAM FOR BEAM OF LENGTH $2a$.

bounding lines OA and OB being such that the angles ZOA, ZOB are $\pm \mu a$. Note that as the angle μa tends to $\pi/2$ the point X tends to infinity, and the circle cannot be drawn. This condition, ($\mu a = \pi/2$), corresponds to a value of P equal to the Euler failing load, and the strut has then become unstable. It is important to realise that if μa is greater than or equal to $\pi/2$, the strut is unstable and the polar diagram has no meaning.

Sign Convention. This is contained in equations 11.13 and 11.14. m is positive when measured upwards from O. The shear at N is positive when N lies to the left of X, negative when N lies to the right of X in the positive quadrant. These conventions are illustrated in Fig. 11.7. If X and N are in the negative quadrant, as X' and N' in Fig. 11.7, the shear is positive when N' lies to the left of X', looking along the radius vector in the positive (upwards) direction. Thus, in the figure, the shear at N' is positive.

The procedure for drawing the polar diagram for different loading cases will now be described, and illustrated by numerical examples.

CASE I—END MOMENTS ONLY, NO LATERAL LOAD (Fig. 11.8)

As we have already seen, in this case $g(x)$ is zero, so that $m = M$, i.e., the difference moment is the true bending moment. We proceed as follows:—Calculate $\mu^2 = P/EI$, and hence find the angle μa , which will, of course, be in

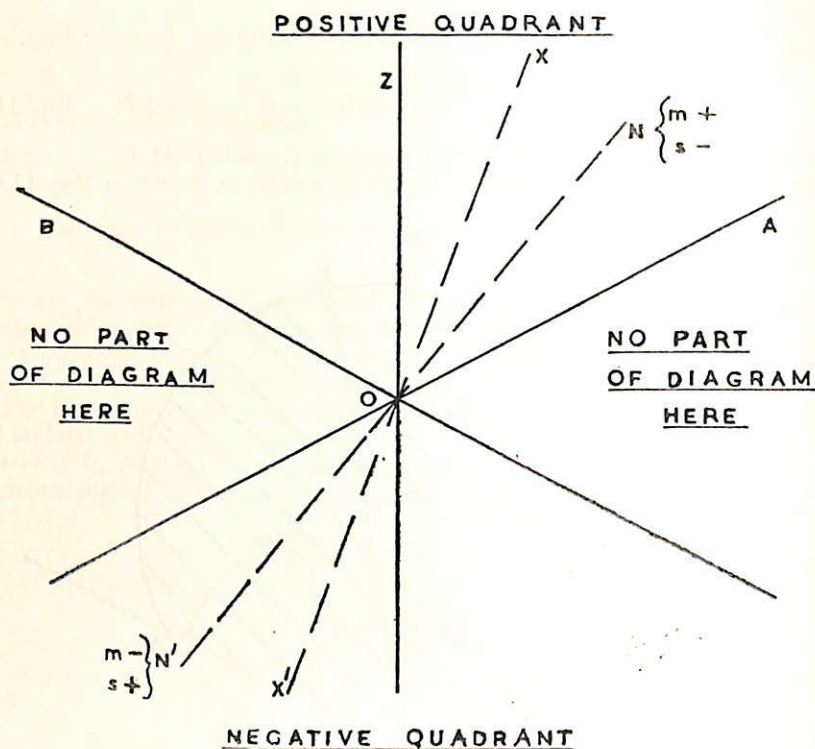


FIG. 11.7.—SIGN CONVENTION FOR DIFFERENCE SHEAR AND DIFFERENCE MOMENT IN POLAR DIAGRAM.

radians. The line OZ (Fig. 11.8 (b)) is drawn as the reference line representing the centre of the beam, and lines OA, OB inclined at $\pm \mu a$ respectively to OZ are drawn to represent the ends of the beam. Distances OA and OB are marked off on these lines, representing m_A and m_B to some convenient scale. In this case $m_A = M_A$, and $m_B = M_B$. We now require the circle passing through O, A and B. If we draw normals to OA and OB from A and B respectively to intersect in X, OX is the diameter of the required circle, and the polar diagram is as shown. The maximum value of m is obviously OX, and the angle ZOx gives the location of this bending moment on the beam.

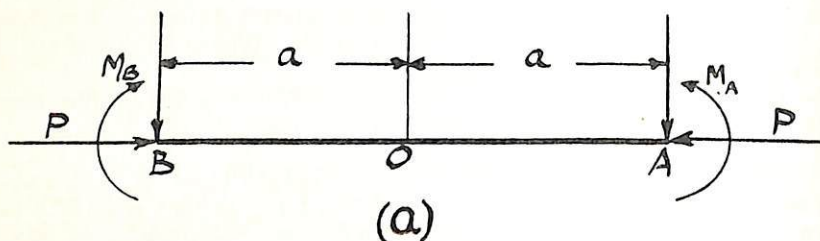


FIG. 11.8 (a).—CASE I. STRUT WITH END MOMENTS.

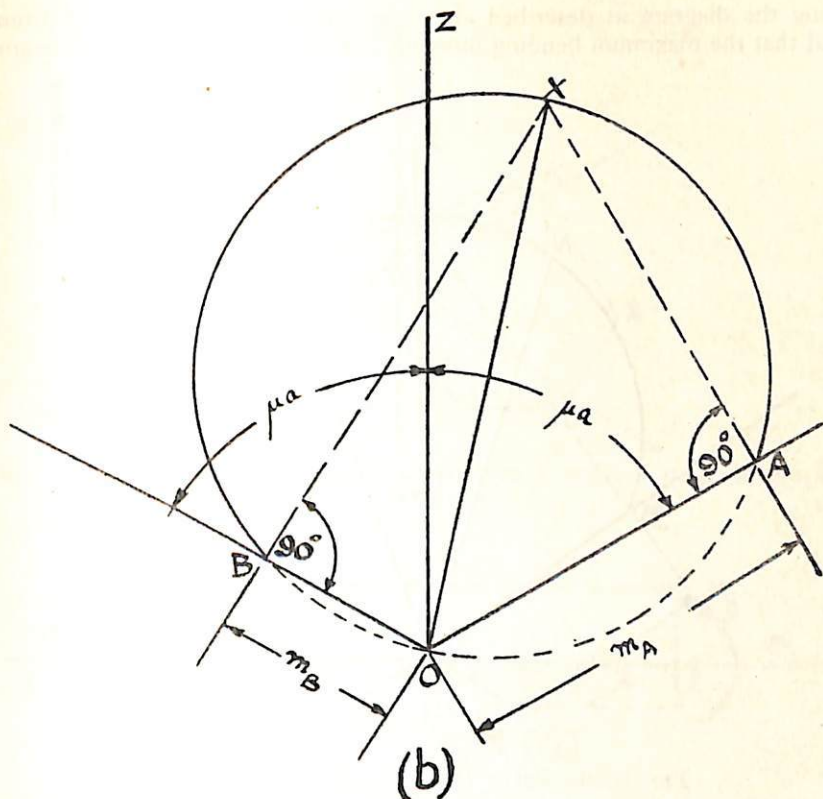


FIG. 11.8 (b).—POLAR DIAGRAM FOR STRUT OF FIG. 11.8 (a).

Example 11.1. For the beam in Fig. 11.9, $EI = 1.2 \times 10^5$ lb. in.² The end load of 500 lb. is offset 0.2 in. from the axis. Draw the Howard polar diagram,

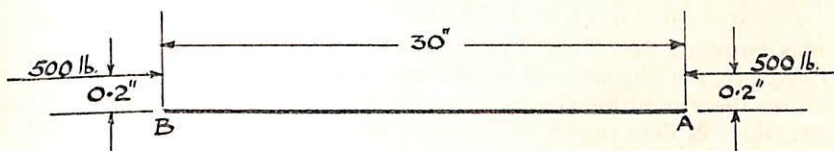


FIG. 11.9.—STRUT FOR EXAMPLE 11.1.

and from it determine the maximum bending moment and the shear at 10 in. from each end. (A.F.R.Ae.S.)

We calculate $m_A = M_A = 500 \times 0.2 = 100$ lb. in.

$m_B = M_B = 500 \times 0.2 = 100$ lb. in.

$$\mu^2 = P/EI = \frac{500}{1.2 \times 10^5} = 0.00417 \text{ in.}^{-2}$$

$$\mu = 0.0645 \text{ in.}^{-1}$$

$$\mu a = 15 \times 0.0645 = 0.968 \text{ radian} = 55.5^\circ$$

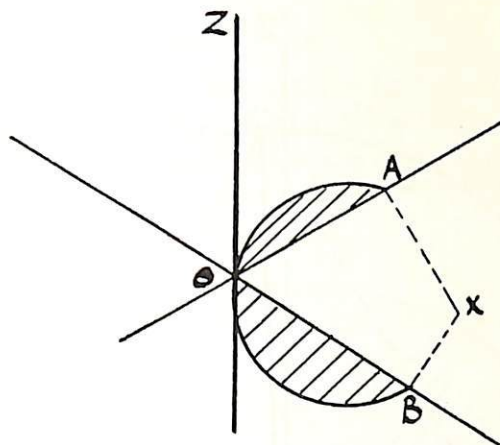


FIG. 11.11.—FORM OF POLAR DIAGRAM FOR CASE I IF M_A IS POSITIVE AND M_B IS NEGATIVE.

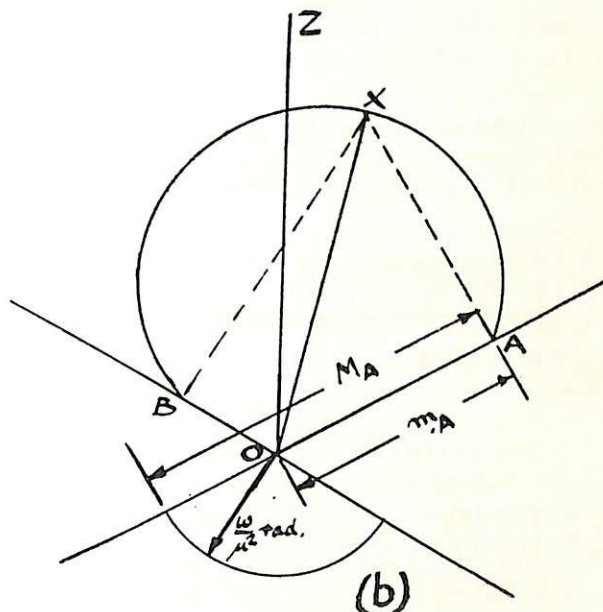
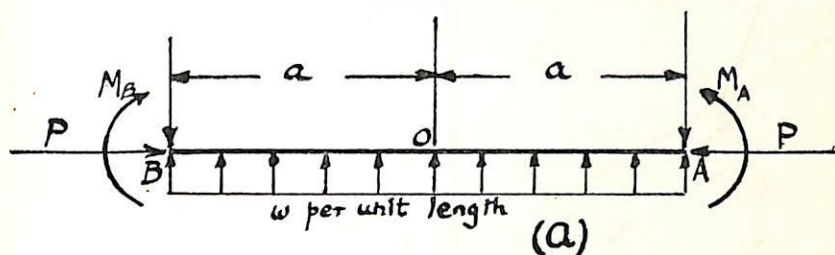
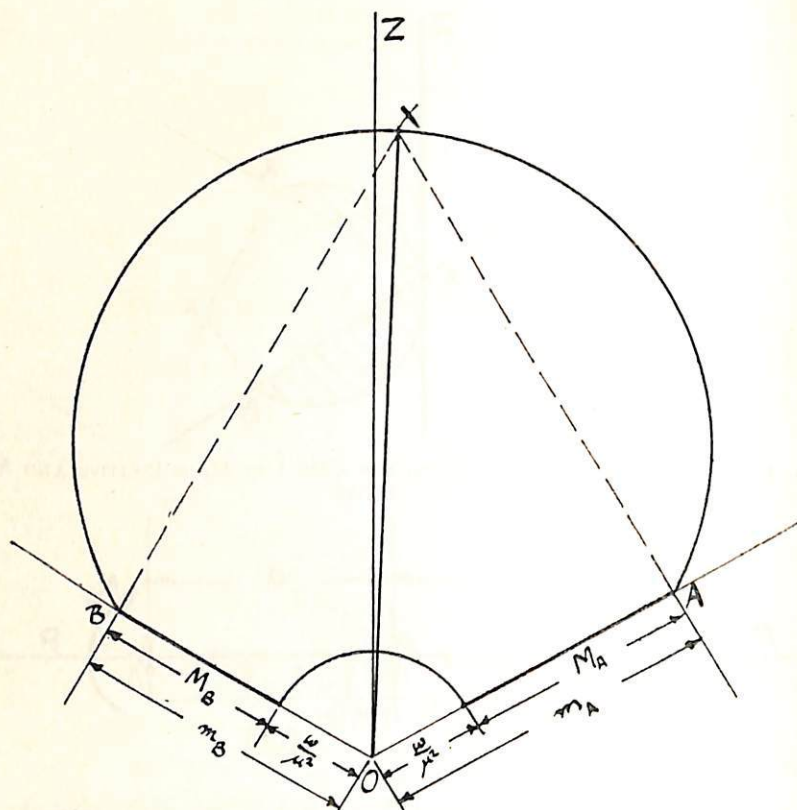


FIG. 11.12.—CASE II. STRUT WITH END MOMENTS AND UNIFORMLY DISTRIBUTED LATERAL LOAD.

FIG. 11.13.—POLAR DIAGRAM FOR CASE II WHEN w IS NEGATIVE.

Example 11.2. For the strut in Fig. 11.14 (a), with end load equal to half of the Euler buckling load, determine, by means of a polar diagram, the distance from the centre of the strut of the point of contraflexure, and find the shear at this point. (A.F.R.Ae.S.)

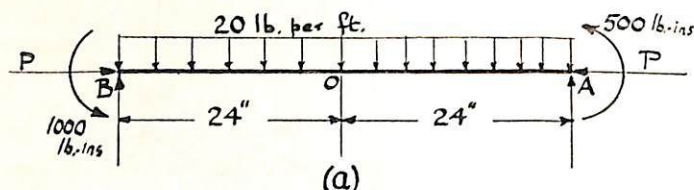


FIG. 11.14 (a). STRUT FOR EXAMPLE 11.2.

$$\mu^2 = \frac{P}{EI} = \frac{P_e}{2EI} = \frac{\pi^2 EI}{2EIL^2} = \frac{\pi^2}{2L^2} = 0.00214 \text{ in.}^{-2}$$

$$\mu = 0.0463 \text{ in.}^{-1}$$

$$\mu a = 0.0463 \times 24 = 1.11 \text{ radians} = 63.6^\circ$$

$$\frac{w}{\mu^2} = \frac{-20/12}{0.00214} = -778 \text{ lb. in.}$$

$$m_A = M_A - w/\mu^2 = 500 + 778 = 1278 \text{ lb. in.}$$

$$m_B = M_B - w/\mu^2 = -1000 + 778 = -222 \text{ lb. in.}$$

The polar diagram, drawn as described above, is shown in Fig. 11.14 (b). Note that, since w is negative, the w/μ^2 circle, A_1CB_1 , is drawn in the positive quadrant.

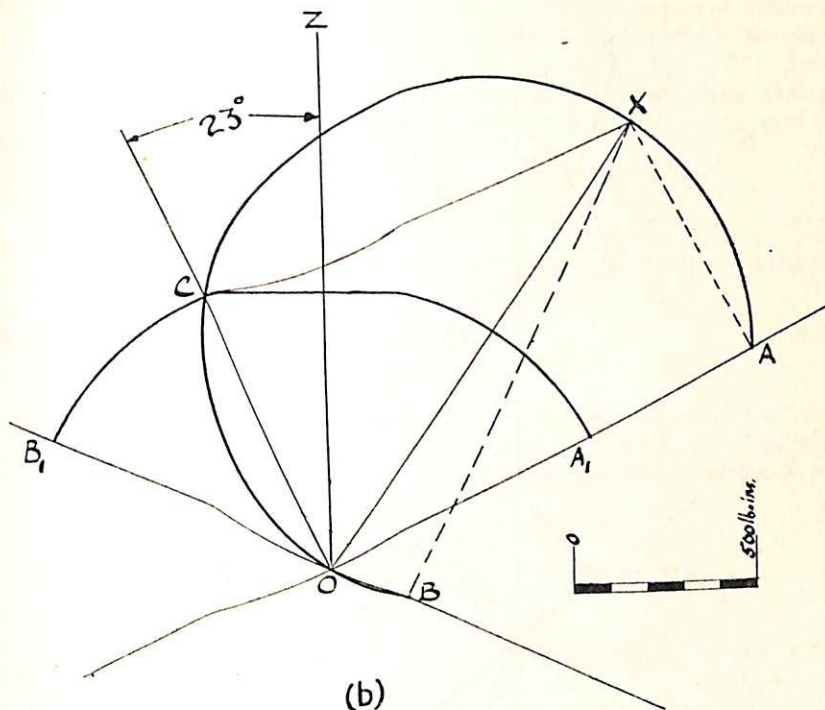


FIG. 11.14 (b).—POLAR DIAGRAM FOR EXAMPLE 11.2.

The point of contraflexure, or point of zero bending moment, is the point C on the diagram, and the location of this point on the beam is given by the angle ZOC, which, from the diagram, is 23° . If this point is at distance x from the centre of the beam, then x is given by

$$\mu x = \frac{23}{57.3} = 0.4014 \text{ radian}$$

$$\text{i.e., } x = \frac{0.4014}{0.0463} = 8.67 \text{ in.}$$

The shear at C = $\mu \cdot XC = 0.0463 \times 1230 = 56.95 \text{ lb.}$

Since C lies to the left of X in the positive quadrant, the shear is positive.

CASE III—END MOMENTS AND CONCENTRATED LOAD (Fig. 11.15)

In view of the discontinuity which must arise in the bending moment at D, we have different values for the constants C and ϵ in equation 11.12 for the two

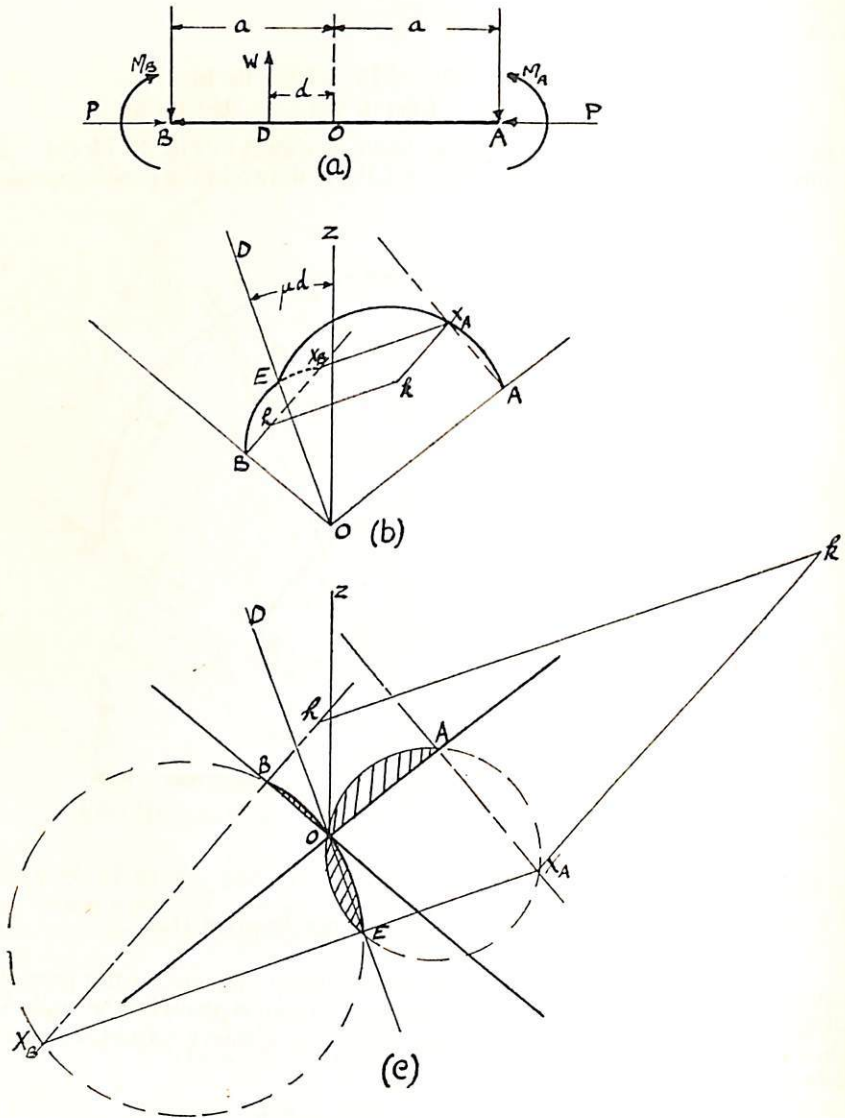


FIG. 11.15.—CASE III. STRUT WITH END MOMENTS AND CONCENTRATED LOAD. (b) FORM OF DIAGRAM IF W/μ IS SMALL; (c) FORM OF DIAGRAM IF W/μ IS LARGE.

portions of the beam AD and DB . Consequently, the polar diagram will contain parts of two circles, one for AD and another for DB , and the problem of construction of the diagram becomes one of determining the apices of the two circles X_A and X_B .

Since, for each portion of the beam, the apparent bending moment, M_1 , will be a linear function of x , containing no powers of x higher than the first, we know

that $g(x)$ will be zero, so that

$$m = M$$

At the point D the bending moment must have the same value whether we consider the portion AD or the portion DB, so that both circles must pass through the same point E on the line OD, which is drawn so that the angle $ZOD = \mu d$ (Fig. 11.15 (b)). It follows that X_A , X_B , and E lie on a straight line perpendicular to OD, the angles OEX_A and OEX_B both being angles in a semi-circle and therefore right angles. As we pass along the beam from B to A, at the point D the apparent shear changes by $+W$. Referring to equation 11.3, we note that, since there is no change in slope as we pass through D, the true shear will also change by $+W$; then from equation 11.14, since in this case $g(x)$ is zero, the difference shear changes by $+W$. Let the difference shear at D as we approach from B be s_{DL} and let this become s_{DR} as we pass to the right of D.

Then

$$s_{DR} = s_{DL} + W$$

But $s_{DL} = \mu \cdot EX_B$ and $s_{DR} = \mu \cdot EX_A$ with due regard to sign. Therefore

$$\mu \cdot EX_A = \mu \cdot EX_B + W$$

or

$$EX_A - EX_B = W/\mu$$

i.e.,

$$X_A X_B = W/\mu$$

Since the shear to the right of D is more positive than the shear to the left of D, the point E must lie further to the left of X_A than of X_B . (Note that this argument refers to relative positions looking in the positive direction along the radius vector.)

We now know the following:—

- (i) the line $X_A X_B$ is at right-angles to OD
- (ii) the distance $X_A X_B$ is W/μ
- (iii) for positive (upwards) W , X_A lies to the right of X_B .

With this knowledge it is possible to locate X_A and X_B .

Having determined the angles μa and μd , draw the lines OZ, OA, OB, OD, marking off distances OA and OB to represent m_A and m_B . Draw the normals to OA and OB from A and B respectively. Then X_A must lie on the normal to OA through A, and X_B must lie on the normal to OB through B. From any point h on the normal to OB, draw a line hk perpendicular to OD, such that the length hk is equal to W/μ measured in the same direction from h as the direction from X_B to X_A (in this case to the right, as in Figs. 11.15 (b) or 11.15 (c)). From k draw a line parallel to Bh to meet the normal to OA in X_A . From X_A draw a line parallel to kh to meet Bh in X_B . Then the points X_A and X_B located in this manner are the required apices, fulfilling the conditions (i), (ii), (iii) above and lying on the appropriate normals. The diagram is completed by drawing the appropriate portions of the circles on OX_A and OX_B as diameters.

A diagram of the type shown in Fig. 11.15 (b) would result if W/μ is sufficiently small. It may well happen that X_A and X_B will lie below OA and OB, in which case the resulting diagram would be similar to that shown in Fig. 11.15 (c).

Example 11.3. A strut, 4 ft. 6 in. long, is pinned at both ends, and carries an end load of $\frac{1}{3}$ of the Euler buckling load. If the end moments are each 1500 lb. in. as shown in Fig. 11.16, and there is a concentrated load of 60 lb. applied at the centre of the strut, determine the maximum bending moment in the strut,

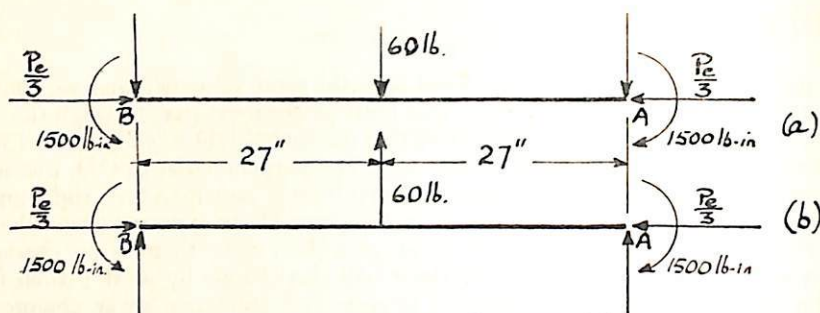


FIG. 11.16.—STRUT FOR EXAMPLE 11.3.

(a) when the 60 lb. load acts downwards

(b) when the 60 lb. load acts upwards. (A.F.R.Ac.S.)

For both cases we may calculate the following.

$$P = \frac{P_e}{3} = \frac{\pi^2 EI}{3L^2}$$

$$\mu^2 = \frac{P}{EI} = \frac{\pi^2}{3L^2} \text{ in.}^{-2}$$

$$\mu = \frac{\pi}{L\sqrt{3}} = 0.0336 \text{ in.}^{-1}$$

$$\mu a = 0.0336 \times 27 = 0.906 \text{ radian} = 51.9^\circ$$

$$m_A = M_A = -1500 \text{ lb. in.}$$

$$m_B = M_B = -1500 \text{ lb. in.}$$

For case (a), $\frac{W}{\mu} = \frac{-60}{0.0336} = -1785 \text{ lb. in.}$

For case (b), $\frac{W}{\mu} = +1785 \text{ lb. in.}$

The respective diagrams are as shown in Figs. 11.17 (a) and 11.17 (b). In case (a), since W/μ is negative, the line hk , starting from a point h on the normal through B, is drawn to the left, and in case (b) this line is drawn to the right.

In case (a), the maximum bending moment occurs at X_A or X_B , and is equal to -1567 lb. in.

In case (b), the maximum bending moment occurs at the centre of the beam, and is equal to -3567 lb. in.

CASE IV—END MOMENTS AND CONCENTRATED MOMENT (Fig. 11.18)

As in Case III, there will be a discontinuity in the bending moment at D, where a concentrated moment, M_D , is applied, so that the polar diagram will contain parts of two circles. The function $g(x)$ is zero, so that $m = M$.

The two apices, X_A and X_B in Fig. 11.18 (b), may be located by the following conditions:—

- (i) There is no change of shear in passing through D, so that X_A and X_B are equidistant from the line OD. Hence the line joining the two apices is parallel to OD.

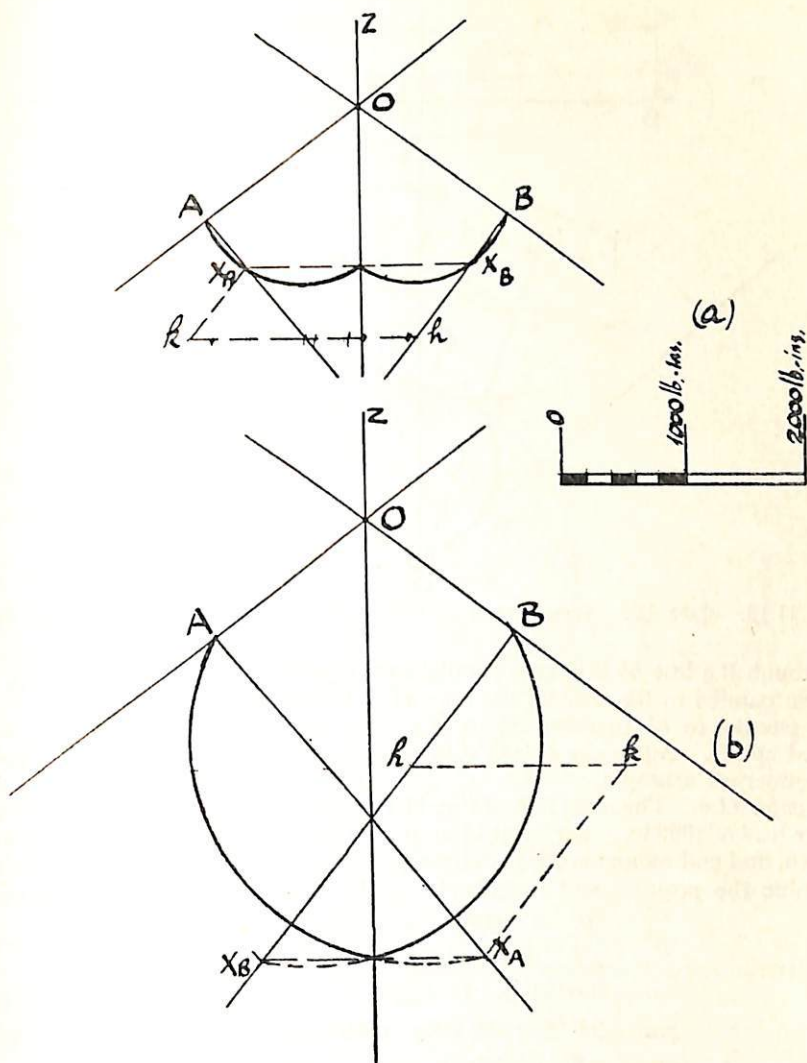


FIG. 11.17.—POLAR DIAGRAMS FOR EXAMPLE 11.3.

- (ii) The bending moment is increased by M_D as we pass through D from B to A , and it follows that m is increased by M_D . Referring to Fig. 11.18 (b), this means that $OE - OF = M_D$. Furthermore, since EX_A and FX_B are both perpendicular to OD and of the same length, because of (i), it follows that $X_A X_B = M_D$.
- (iii) X_A and X_B lie respectively on the normal to OA from A and the normal to OB from B .

The location of the apices can be determined from these conditions by a construction similar to that used in Case III. From any point h on the normal to

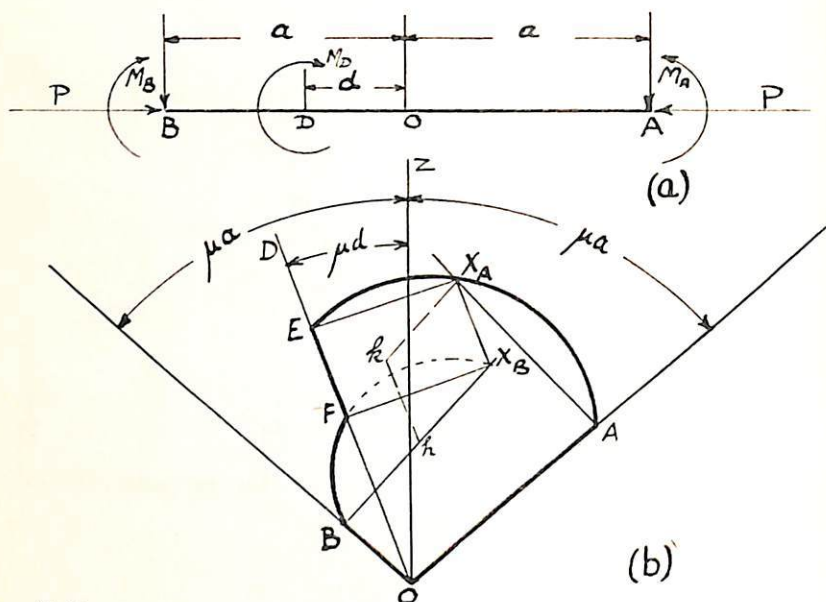


FIG. 11.18.—CASE IV. STRUT WITH END MOMENTS AND CONCENTRATED MOMENT.

OB through B a line hk is drawn parallel to OD and of length equal to M_D . kX_A is drawn parallel to Bh to meet the normal to OA through A in X_A , and X_AX_B is drawn parallel to kh to meet Bh in X_B . X_A and X_B thus determined are the required apices. Again the practical application of the construction is illustrated by a numerical example.

Example 11.4. The strut AB, of Fig. 11.19 (a), is 46.3 in. long and carries a compressive load of 8000 lb. At a point 15 in. from B a moment of 1000 lb. in. is applied as shown, and end moments are as indicated in the diagram. $EI = 5.4 \times 10^6$ lb. in.² Determine the position and magnitude of the maximum bending moment.

$$\mu^2 = \frac{P}{EI} = \frac{8000}{5.4 \times 10^6} = 0.001482 \text{ in.}^{-2}$$

$$\mu = 0.0385 \text{ in.}^{-1}$$

$$\mu a = 23.15 \times 0.0385 = 0.892 \text{ radian} = 51.1^\circ$$

$$m_A = M_A = 3000 \text{ lb. in.}$$

$$m_B = M_B = 4000 \text{ lb. in.}$$

$$\mu d = 0.0385 \times 10 = 0.385 \text{ radian} = 22.05^\circ$$

The diagram, constructed as described above, is shown in Fig. 11.19 (b). Note that hk is drawn in the direction of increasing positive bending moment, since the bending moment is increased by the concentrated moment as we pass through D from B to A. The maximum bending moment is equal to $OX_A = 6416$ lb. in.

The angle $ZOX_A = 11^\circ = 0.192$ radian. Hence, the distance, x , of the point of maximum bending moment from the centre is given by

$$\mu x = 0.192$$

i.e.,

$$x = 4.98 \text{ in.}$$

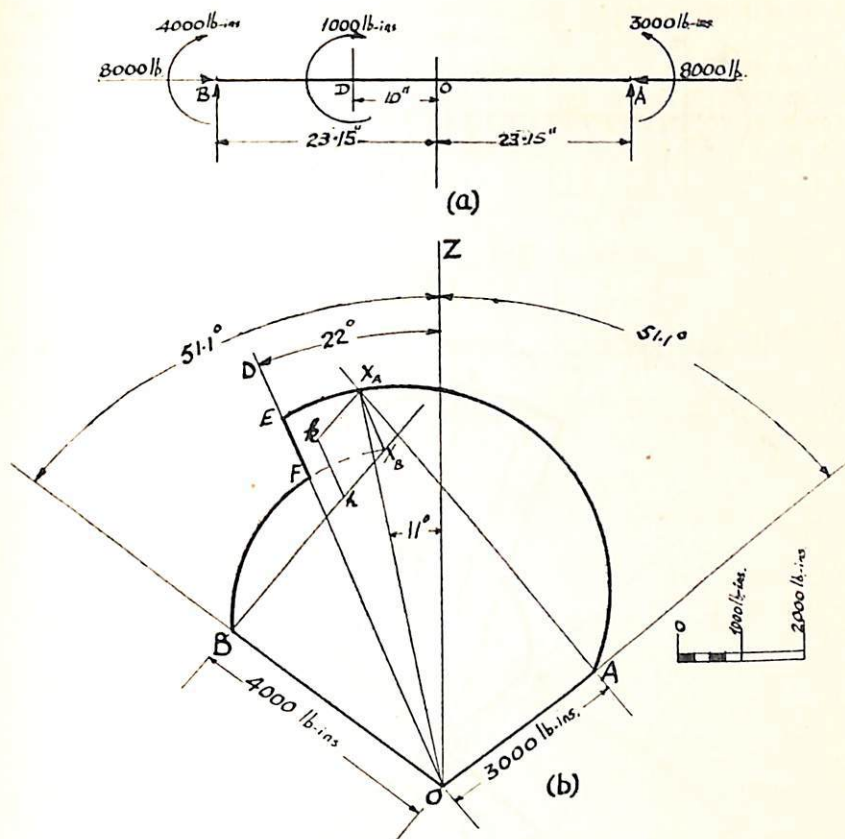


FIG. 11.19.—EXAMPLE 11.4.

CASE V—END MOMENTS AND SUDDEN CHANGE IN UNIFORMLY DISTRIBUTED LOAD (Fig. 11.20)

Over the portion BD the distributed load is w_1 per unit length, and over DA it is w_2 per unit length. For the purposes of the typical diagram in Fig. 11.20 (b), it is assumed that w_1 is greater than w_2 .

We know that $m = M - w/\mu^2$, so that

$$\text{for BD} \quad m = M - w_1/\mu^2$$

$$\text{and for DA} \quad m = M - w_2/\mu^2$$

It is obvious that we shall again have two different circles for the two portions BD and DA with corresponding apices X_B and X_A . At the point D there is no change in shear, so that X_B and X_A are equidistant from OD in the polar diagram, and $X_B X_A$ is parallel to OD. Furthermore, at D there is no change in the bending moment as we pass from one side of D to the other, so that we may write

$$M_{DB} = M_{DA}$$

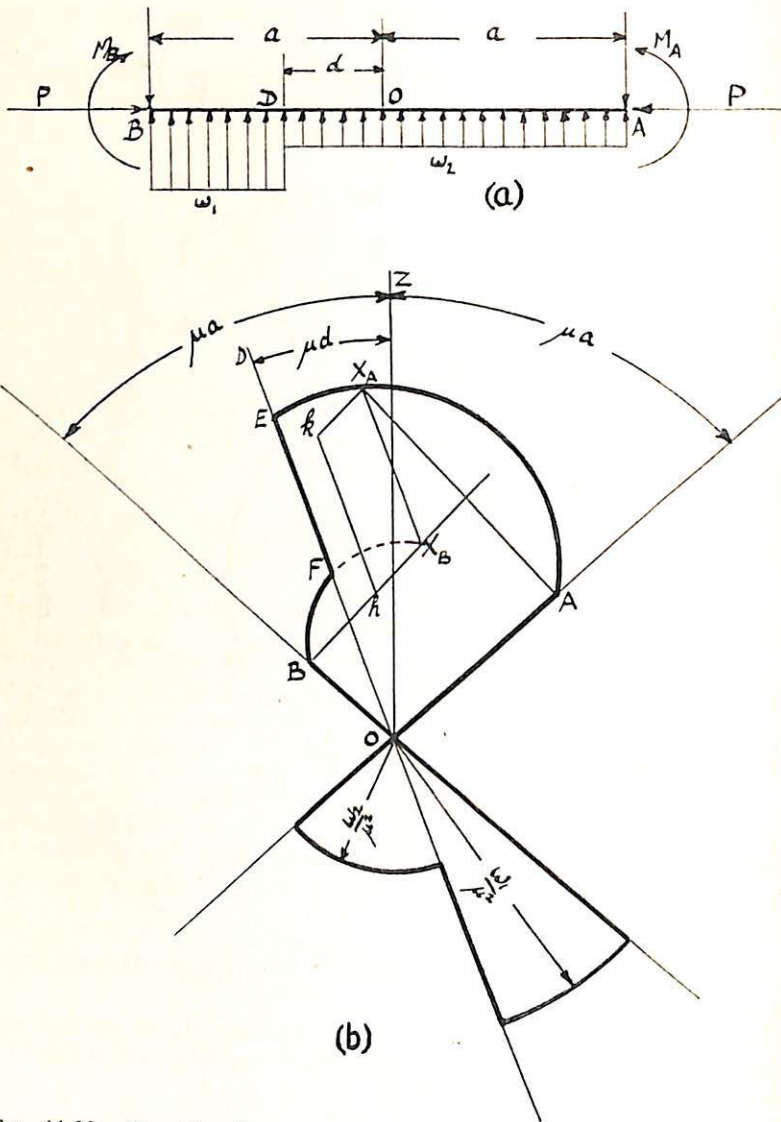


FIG. 11.20.—CASE V. STRUT WITH END MOMENTS AND SUDDEN CHANGE IN UNIFORMLY DISTRIBUTED LOAD.

Hence,

$$m_{DB} + w_1/\mu^2 = m_{DA} + w_2/\mu^2$$

giving

$$m_{DA} - m_{DB} = \frac{w_1 - w_2}{\mu^2}$$

Reference to Fig. 11.20 (b) shows that $m_{DA} - m_{DB} = OE - OF = X_A X_B$. Since the apices are located on the normals to OA and OB through A and B respectively, it is now possible to determine their positions.

The construction follows that of Case IV, the length of the line hk in this instance being $(w_1 - w_2)/\mu^2$.

Example 11.5. A strut, 10 ft. long, simply supported at each end, is under the action of a compressive force of 1000 lb. One half carries a transverse load of 50 lb. per ft., and the other half carries a load of 150 lb. per ft. Draw the Howard Polar diagram for the bending moment, and hence draw a shear force diagram. Take $EI = 3 \times 10^6$ lb. in.² (A.F.R.Ae.S.)

In this case M_A and M_B are both zero.

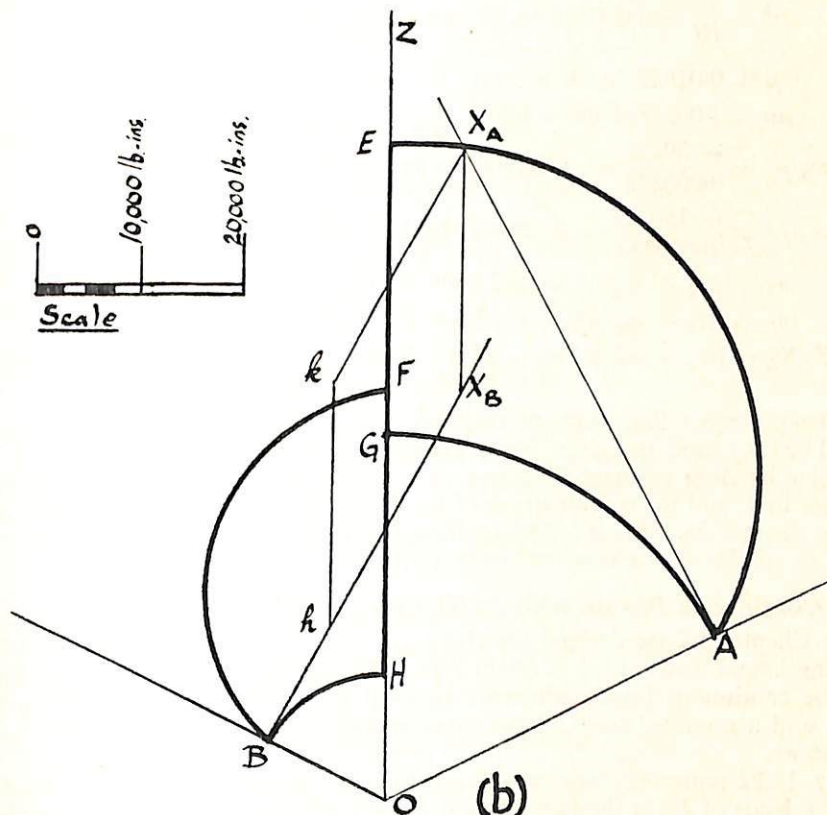
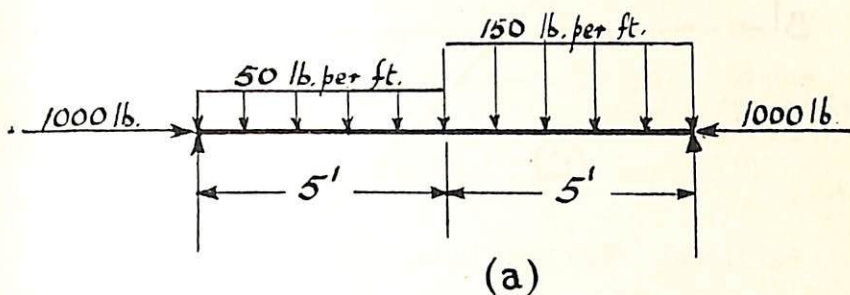


FIG. 11.21(a) and (b).—EXAMPLE 11.5.

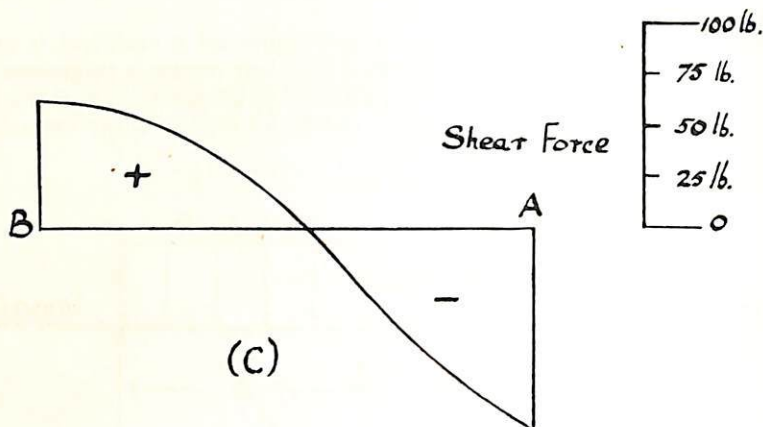


FIG. 11.21 (c).—SHEAR FORCE DIAGRAM FOR STRUT OF EXAMPLE 11.5.

$$\mu^2 = \frac{P}{EI} = 0.000333 \text{ in.}^{-2}$$

$$\mu = 0.01827 \text{ in.}^{-1}$$

$$\mu a = 1.095 \text{ radians} = 62.75^\circ$$

$$w_1/\mu^2 = \frac{-50/12}{0.000333} = -12,500 \text{ lb. in.}$$

$$w_2/\mu^2 = \frac{-150/12}{0.000333} = -37,500 \text{ lb. in.}$$

$$m_A = M_A - w_2/\mu^2 = +37,500 \text{ lb. in.}$$

$$m_B = M_B - w_1/\mu^2 = +12,500 \text{ lb. in.}$$

$$X_B X_A = (w_1 - w_2)/\mu^2 = +25,000 \text{ lb. in.} \quad (\text{The } + \text{ sign indicates that } X_A \text{ is above } X_B.)$$

The polar bending moment diagram, constructed as described above, is in Fig. 11.21 (b), and the shear force diagram is in Fig. 11.21 (c).

Polar bending moment diagrams may be drawn also for struts with sudden changes in I , and for combinations of the types of loading considered in the preceding paragraphs. For a complete discussion of polar diagrams, reference should be made to Howard's original paper (*A.R.C. R. O & E M. 1233*).

11.4. Continuous Beams with Axial Compression

In Chapter IX we derived the three moment equation for continuous beams carrying lateral loads only. Certain types of wing spars, and fuselage longerons, may be continuous beams subjected to axial compression in addition to lateral loads, and a modified form of the three moment equation may be obtained for such cases.

Fig. 11.22 represents two bays of a continuous beam, which carries axial compressive loads of P_1 in the bay AB and P_2 in the bay BC, in addition to the distributed lateral loads. We consider the bay AB, the loading on which may be

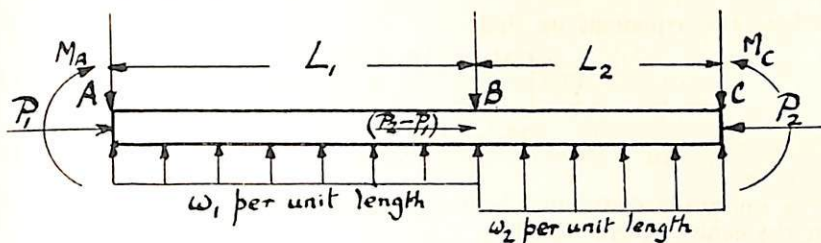


FIG. 11.22.—CONTINUOUS BEAM WITH AXIAL COMPRESSION.

represented as shown in Fig. 11.23. If we denote the *apparent* bending moment (neglecting the end load) as M_1 , we may write for the total bending moment at a point, distant x from the centre of the beam, where the deflection is y ,

$$M = M_1 - P_1 y$$

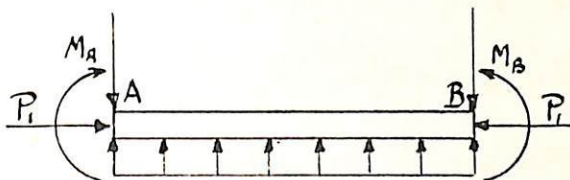


FIG. 11.23.—LOADING ON BAY AB OF BEAM SHOWN IN FIG. 11.22.

Differentiating this equation twice gives

$$\frac{d^2 M}{dx^2} = \frac{d^2 M_1}{dx^2} - P_1 \frac{d^2 y}{dx^2}$$

and we know that $\frac{d^2 M_1}{dx^2} = w_1$ = intensity of loading. Hence, using $\frac{d^2 y}{dx^2} = \frac{M}{EI}$,

and putting $\mu_1^2 = \frac{P_1}{EI}$, we have

$$\frac{d^2 M}{dx^2} + \mu_1^2 M = w_1 \quad \text{Eq. 11.15}$$

w_1 may or may not be constant along the length of the bay. We shall consider for simplicity the case in which the lateral load is uniformly distributed over the bay, so that w_1 is constant. Then the solution to equation 11.15 is of the form

$$M = A \cos \mu_1 x + B \sin \mu_1 x + \frac{w_1}{\mu_1^2} \quad \text{Eq. 11.16}$$

where A and B are constants.

When $x = -L_1/2$, we know that $M = M_A$

and when $x = +L_1/2$, we know that $M = M_B$

Hence, from equation 11.16, writing $\mu_1 L_1/2 = \alpha_1$

$$M_A = A \cos \alpha_1 - B \sin \alpha_1 + \frac{w_1}{\mu_1^2} \quad \text{Eq. 11.17}$$

$$M_B = A \cos \alpha_1 + B \sin \alpha_1 + \frac{w_1}{\mu_1^2} \quad \text{Eq. 11.18}$$

From these two equations we find

$$A = \left[\frac{M_A + M_B}{2} - \frac{w_1}{\mu_1^2} \right] \sec \alpha_1 \quad \text{Eq. 11.19}$$

$$B = \frac{M_B - M_A}{2} \operatorname{cosec} \alpha_1 \quad \text{Eq. 11.20}$$

For a uniformly distributed load w_1 , the apparent bending moment M_1 , at x from the centre of AB will be

$$M_1 = M_B + \frac{w_1}{2} \left(\frac{L_1}{2} - x \right)^2 - \frac{w_1 L_1}{2} \left(\frac{L_1}{2} - x \right) - \frac{M_B - M_A}{L_1} \left(\frac{L_1}{2} - x \right) \quad \text{Eq. 11.21}$$

From our initial equation, $M = M_1 - P_1 y$, together with equations 11.16 and 11.21, we have

$$P_1 y = M_B + \frac{w_1}{2} \left(\frac{L_1}{2} - x \right)^2 - \left(\frac{w_1 L_1}{2} + \frac{M_B - M_A}{L_1} \right) \left(\frac{L_1}{2} - x \right) - A \cos \mu_1 x - B \sin \mu_1 x - \frac{w_1}{\mu_1^2}$$

Hence,

$$P_1 \frac{dy}{dx} = -w_1 \left(\frac{L_1}{2} - x \right) + \left(\frac{w_1 L_1}{2} + \frac{M_B - M_A}{L_1} \right) + \mu_1 A \sin \mu_1 x - \mu_1 B \cos \mu_1 x$$

When $x = L_1/2$, we have $\frac{dy}{dx} = \text{slope at } B = i_B$

Thus,

$$P_1 i_B = \frac{w_1 L_1}{2} + \frac{M_B - M_A}{L_1} + \mu_1 A \sin \mu_1 \frac{L_1}{2} - \mu_1 B \cos \mu_1 \frac{L_1}{2}$$

Putting $P_1 = \mu_1^2 EI$, and inserting values of A and B from equations 11.19 and 11.20, noting also that $\mu_1 \frac{L_1}{2}$ is denoted by α_1 , this becomes

$$\begin{aligned} EI i_B &= \frac{M_A}{\mu_1^2} \left[\frac{-1}{L_1} + \frac{\mu_1 \tan \alpha_1}{2} + \frac{\mu_1 \cot \alpha_1}{2} \right] + \frac{M_B}{\mu_1^2} \left[\frac{1}{L_1} + \frac{\mu_1 \tan \alpha_1}{2} - \frac{\mu_1 \cot \alpha_1}{2} \right] \\ &\quad - \frac{w_1}{\mu_1^2} \left[-\frac{L_1}{2} + \frac{\tan \alpha_1}{\mu_1} \right] \\ &= \frac{M_A}{\mu_1} \left[-\frac{1}{2\alpha_1} + \frac{\tan \alpha_1}{2} + \frac{\cot \alpha_1}{2} \right] + \frac{M_B}{\mu_1} \left[\frac{\tan \alpha_1}{2} - \frac{\cot \alpha_1}{2} + \frac{1}{2\alpha_1} \right] \\ &\quad - \frac{w_1 L_1}{\mu_1^2} \left[-\frac{1}{2} + \frac{\tan \alpha_1}{2\alpha_1} \right] \\ &= \frac{M_A L_1}{4} \left[\frac{2\alpha_1 \operatorname{cosec} 2\alpha_1 - 1}{\alpha_1^2} \right] + \frac{M_B L_1}{4} \left[\frac{1 - 2\alpha_1 \cot 2\alpha_1}{\alpha_1^2} \right] \\ &\quad + \frac{w_1 L_1^3}{8} \left[\frac{\alpha_1 - \tan \alpha_1}{\alpha_1^3} \right] \end{aligned}$$

Writing $L_1 = 2\alpha_1$, we have

$$\begin{aligned} EI i_B &= \frac{M_A a_1}{2} \left[\frac{2\alpha_1 \operatorname{cosec} 2\alpha_1 - 1}{\alpha_1^2} \right] + \frac{M_B a_1}{2} \left[\frac{1 - 2\alpha_1 \cot 2\alpha_1}{\alpha_1^2} \right] \\ &\quad + w_1 a_1^3 \left[\frac{\alpha_1 - \tan \alpha_1}{\alpha_1^3} \right] \end{aligned}$$

Similarly, for the bay BC,

$$-EIi_B = \frac{M_C a_2}{2} \left[\frac{2\alpha_2 \operatorname{cosec} 2\alpha_2 - 1}{\alpha_2^2} \right] + \frac{M_B a_2}{2} \left[\frac{1 - 2\alpha_2 \cot 2\alpha_2}{\alpha_2^2} \right] + w_2 a_2^3 \left[\frac{\alpha_2 - \tan \alpha_2}{\alpha_2^3} \right]$$

Eliminating i_B from these two equations,

$$\begin{aligned} & \frac{M_A a_1}{2} \left[\frac{2\alpha_1 \operatorname{cosec} 2\alpha_1 - 1}{\alpha_1^2} \right] + \frac{M_B}{2} \left[a_1 \left\{ \frac{1 - 2\alpha_1 \cot 2\alpha_1}{\alpha_1^2} \right\} + a_2 \left\{ \frac{1 - 2\alpha_2 \cot 2\alpha_2}{\alpha_2^2} \right\} \right] \\ & + \frac{M_C a_2}{2} \left[\frac{2\alpha_2 \operatorname{cosec} 2\alpha_2 - 1}{\alpha_2^2} \right] + w_1 a_1^3 \left[\frac{\alpha_1 - \tan \alpha_1}{\alpha_1^3} \right] + w_2 a_2^3 \left[\frac{\alpha_2 - \tan \alpha_2}{\alpha_2^3} \right] = 0 \end{aligned}$$

Multiplying this equation throughout by 3, we may write the three moment equation for a beam subjected to axial compressive loads and uniformly distributed lateral loads in the form

$$M_A a_1 f(\alpha_1) + 2M_B \{a_1 \phi(\alpha_1) + a_2 \phi(\alpha_2)\} + M_C a_2 f(\alpha_2) + w a_1^3 \psi(\alpha_1) + w a_2^3 \psi(\alpha_2) = 0 \quad \text{Eq. 11.22}$$

where $f(\alpha)$, $\phi(\alpha)$, $\psi(\alpha)$ are the *Berry functions*, defined as

$$f(\alpha) = \frac{3}{2} \left\{ \frac{2\alpha \operatorname{cosec} 2\alpha - 1}{\alpha^2} \right\}$$

$$\phi(\alpha) = \frac{3}{4} \left\{ \frac{1 - 2\alpha \cot 2\alpha}{\alpha^2} \right\}$$

$$\psi(\alpha) = 3 \left\{ \frac{\alpha - \tan \alpha}{\alpha^3} \right\}$$

These functions are so named after their use by A. Berry, who investigated the foregoing problem (*Transactions of the Royal Aeronautical Society*, No. 1).

The theory may be extended to cover the case where the supports are not all at the same level, and, of course, the case where the value of EI is not the same for both bays.

11.5. Continuous Beams with Axial Tension

If the axial loads in the several bays are tensile instead of compressive, an equation of the same form as equation 11.22 still holds, if the functions $f(\alpha)$, $\phi(\alpha)$, $\psi(\alpha)$ are replaced by functions $F(\alpha)$, $\Phi(\alpha)$, $\Psi(\alpha)$ defined by

$$F(\alpha) = \frac{3}{2} \left\{ \frac{2\alpha \operatorname{cosech} 2\alpha - 1}{\alpha^2} \right\}$$

$$\Phi(\alpha) = \frac{3}{4} \left\{ \frac{2\alpha \coth 2\alpha - 1}{\alpha^2} \right\}$$

$$\Psi(\alpha) = 3 \left\{ \frac{\alpha - \tanh \alpha}{\alpha^3} \right\}$$

TENSION FIELD BEAMS

12.1. Introduction

In para. 5.10, it was mentioned that a thin web may buckle under shear into diagonal waves. The analysis of a beam with a web which behaves in this manner was presented in detail by H. Wagner in 1929, and much work has followed on the development of the theory.

It should be realised that the buckling of the web does not render the beam unsafe, and that it is not indicative of failure of the beam. Up to a certain point the buckles will disappear on removal of the load. In practice, of course, the web will not buckle before the shear reaches a certain critical value, and later theories allow for the fact that the diagonal tension fields formed are not necessarily *complete*, the web continuing to take a proportion of shear beyond the critical value in the normal way (i.e., by diagonal tension *and* compression), instead of taking the entire shear beyond the critical value by means of diagonal tension only.

We shall not attempt to discuss in this book the later developments in detail, and we shall confine our theoretical investigation to the complete tension field, in which we assume that, when the web has once buckled, it can support no further diagonal compression at all. Indeed, for simplicity in the development of our arguments, we may make the assumption that the web buckles immediately, so that the diagonal compressive stress must be zero. We could say, alternatively, that the shear force, S , considered in the following paragraphs is the total shear force less the shear carried up to the buckling shear, so that the effects derived are those due solely to the part of the total shear which is carried by tension fields.

12.2. The Complete Tension Field Web

In the beam of Fig. 12.1 (a), let the distance between the vertical stiffeners be b ,

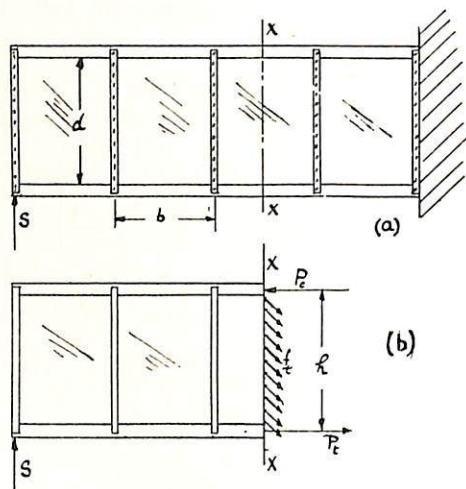


FIG. 12.1.—TENSION FIELD BEAM.

the depth of the web d , the distance between the centroids of the upper and lower flange members h , and the thickness of the web t .

We assume that the web carries the whole of the shear force, S , by diagonal tension. Let the tensile stress so produced in the web be f_t and let the direction of this tensile stress be at an angle α to the horizontal, as shown. Considering a portion of the beam between the tip and section XX, the forces producing equilibrium are the applied shear, S , the tension in the web, and the end loads in the flanges, P_c and P_t , as indicated in Fig. 12.1 (b).

EVALUATION OF WEB TENSILE STRESS

Fig. 12.2 represents a triangular element of the web, in which AB is the depth of the web, d , BC is at right-angles to the tension lines and AC is parallel to them.

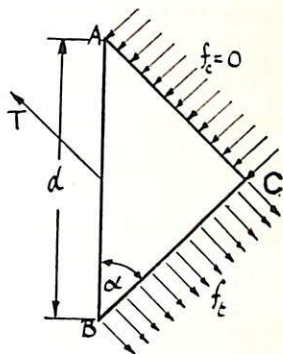


FIG. 12.2.—FORCES ON ELEMENT OF TENSION FIELD WEB.

The internal forces on this piece of web may be represented as shown, the compressive stress on AC being zero in accordance with our initial assumption. The length $BC = AB \cos \alpha = d \cos \alpha$, so that the area of the cross-section BC is equal to $dt \cos \alpha$.

Hence,
$$T = f_t dt \cos \alpha \quad \text{Eq. 12.1}$$

Referring to Fig. 12.1 (b), or to Fig. 12.3, we see, by resolving forces vertically, that

$$T \sin \alpha = S \quad \text{Eq. 12.2}$$

From equations 12.1 and 12.2, we obtain

$$f_t = \frac{S}{dt \sin \alpha \cos \alpha} \quad \text{Eq. 12.3}$$

But $\frac{S}{dt} = f_s = \text{apparent or nominal shear stress}$

$$\text{i.e.,} \quad f_t = \frac{f_s}{\sin \alpha \cos \alpha} = \frac{2f_s}{\sin 2\alpha} \quad \text{Eq. 12.4}$$

END LOADS IN FLANGE MEMBERS

If, referring to Fig. 12.3, we take moments about the point marked A, we have

$$Sx - P_t h - T \cos \alpha \cdot \frac{d}{2} = 0$$

$$\text{i.e.,} \quad P_t = \frac{Sx}{h} - \frac{T \cos \alpha \cdot d}{2h}$$

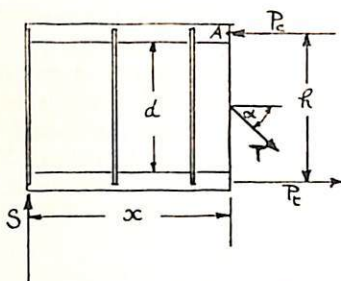


FIG. 12.3.—END LOADS IN FLANGES OF TENSION FIELD BEAM.

Substituting for T from equation 12.2, we obtain

$$P_t = \frac{Sx}{h} - \frac{S \cot \alpha \cdot d}{2h}$$

Since, in general, d will be approximately equal to h , this may be written

$$P_t = \frac{Sx}{h} - \frac{S \cot \alpha}{2} \quad \text{Eq. 12.5}$$

Putting $Sx = M$, the bending moment at the section considered,

$$P_t = \frac{M}{h} - \frac{S \cot \alpha}{2} \quad \text{Eq. 12.6}$$

Similarly,

$$P_c = \frac{M}{h} + \frac{S \cot \alpha}{2} \quad \text{Eq. 12.7}$$

It appears from these equations that one effect of the formation of tension fields in the web is to produce an additional compressive end load in each flange, of amount $S \cot \alpha / 2$.

Equations 12.6 and 12.7 may be derived in a different manner, which emphasises the physical aspects of the problem. First, consider the length x of the upper flange. As indicated in Fig. 12.4, the web tension will apply a force to this length

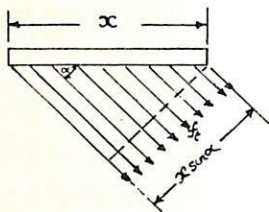


FIG. 12.4.—LOADING OF FLANGE BY DIAGONAL TENSION IN WEB.

of flange equal to $(f_t x t \sin \alpha)$, parallel to the tension lines. The horizontal component of this force

$$= (f_t x t \sin \alpha) \cos \alpha$$

$$= \frac{Sx}{d} \quad (\text{from equation 12.3})$$

$$= \frac{M}{h}, \text{ taking } d \simeq h, \text{ and } Sx = M$$

Now consider the vertical stiffener at the free end of the beam. As shown in Fig. 12.5, there will be a horizontal force on this member due to the unbalanced

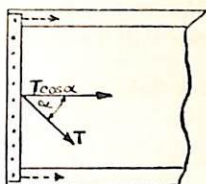


FIG. 12.5.—AXIAL COMPRESSION IN FLANGES.

horizontal component of the web tension which must produce equal compressive loads in the two flange members. This horizontal force is equal to $T \cos \alpha$, so that the compressive force in each flange arising from this is $\frac{1}{2}T \cos \alpha$. Substituting for T from equation 12.2, we may write this as $\frac{S}{2} \cot \alpha$.

Hence the total compressive force in the upper flange at section XX is equal to $\frac{M}{h} + \frac{S}{2} \cot \alpha$, and the total tensile force in the lower flange at this section is $\frac{M}{h} - \frac{S}{2} \cot \alpha$, giving the same results as equations 12.6 and 12.7.

LOAD IN VERTICAL STIFFENER

Referring again to Fig. 12.4, there will be a vertical component of the web tension applied to the length x of the flange. This will be a distributed load, the magnitude of which, if we assume uniform distribution, is $(f_t \sin^2 \alpha)$ per unit length. This load is directed inwards on each flange and tends to pull the flanges together. The vertical stiffeners, therefore, serve the purpose of holding the flanges apart, and will consequently be subjected to compression. Assuming that the stiffeners are spaced at distance b apart, the compressive load on an intermediate stiffener is seen to be

$$\begin{aligned} V &= f_t \sin^2 \alpha \cdot b \\ &= \frac{Sb}{d} \tan \alpha \quad \dots \dots \dots \text{Eq. 12.8} \end{aligned}$$

SAGGING BENDING OF FLANGES

The component of the web tension normal to the flanges, which we have just considered in determining the load in the vertical stiffeners, obviously causes *secondary* bending of the flanges themselves, as opposed to the *primary* bending of the beam as a whole. Since this effect tends to cause the flanges to "sag" between stiffeners, it is often called *sagging bending*. Assuming again that the loading causing this sagging bending is uniformly distributed, and regarding the flanges as being supported at the stiffeners, we obtain the maximum bending moment on a flange at a stiffener, the magnitude being $\frac{wb^2}{12}$, where w is the intensity of loading. In this instance, $w = \frac{S \tan \alpha}{d}$, so that the maximum sagging bending moment, M_F , on the flange is given by

$$M_F = \frac{Sb^2 \tan \alpha}{12d} \quad \dots \dots \dots \text{Eq. 12.9}$$

The stresses arising from this bending moment must be added to those due to the end loads already derived. If the flanges have cross-sectional areas A_F , and if the compressive and tensile section moduli of the flanges are Z_c and Z_t respectively, the maximum stresses will be,

$$\text{in compression (upper) flange, } f_c = \frac{P_c}{A_F} + \frac{Sb^2 \tan \alpha}{12dZ_c} \quad \text{Eq. 12.10}$$

$$\text{in tension (lower) flange, } f_t = \frac{P_t}{A_F} + \frac{Sb^2 \tan \alpha}{12dZ_t} \quad \text{Eq. 12.11}$$

ANGLE OF TENSION FIELD

If the flanges are infinitely rigid, the value of α is 45° , and in our elementary discussion this value will be assumed. In practice, α is usually less than 45° , since the ideal conditions assumed are not attained, and a value of 42° is often used in initial rough design. The angle depends largely on the stiffness of the edge members, and on the panel dimensions.

Using $\alpha = 45^\circ$, the following results are obtained,

$$\text{Tensile stress in web} = f_t = \frac{2S}{dt} \quad \text{Eq. 12.12}$$

$$\text{Compressive load in stiffener} = V = \frac{Sb}{d} \quad \text{Eq. 12.13}$$

$$\text{Compressive stress in upper flange} = \frac{\frac{M}{h} + \frac{S}{2}}{A_F} + \frac{Sb^2}{12dZ_c} \quad \text{Eq. 12.14}$$

$$\text{Tensile stress in lower flange} = \frac{\frac{M}{h} - \frac{S}{2}}{A_F} + \frac{Sb^2}{12dZ_t} \quad \text{Eq. 12.15}$$

12.3. Practical Considerations

The above discussion does not give results which are generally applicable to practical cases. However, the theoretical treatment of the complete tension field was the basis for all the later developments in the theory of the incomplete tension field. Much has been written and published on the subject, and numerous methods, both theoretical and empirical, have been suggested for the analysis of beams of this type. No attempt can be made here to present any of these methods. Current practice in this country is summarised in the Royal Aeronautical Society's Data Sheets for stressed skin structures.

Without any attempt at full discussion, it is possible to indicate some of the major items which render the simple theory of the previous paragraphs inapplicable to practical cases, and which have been given much attention in the development of the later theory. We have mentioned already that (a) the web carries some shear before buckling, and that (b) the web continues to support an increasing diagonal compression after buckling, although the compression increases at a slower rate than the diagonal tension, so that not all the shear after buckling is carried by tension fields. The value of the shear stress at which the web will buckle is considered briefly in para. 13.5.

In determining the loads in the vertical stiffeners, and the sagging bending moment on the flanges, we assumed that the tension in the web produced a uniformly distributed load on the flanges. This assumption is justified only if the

flanges are infinitely rigid. The distribution and the magnitude of the lateral loading on the flanges are affected greatly by the stiffness of the flanges, and it follows that the stiffener load is similarly affected.

We have said that the stiffeners serve the purpose of holding the flanges apart, acting as struts in doing so. They have another function, however, in that they stabilise the web. Not only must the stiffeners be capable of carrying the compressive loads imposed on them, they must also have sufficient lateral stiffness to prevent buckling of the web and stiffeners as a whole. As pointed out in para. 13.4, part of the web adjacent to a stiffener will work with the stiffener in carrying the compressive load, and due account should be taken of this.

The latest available results take the above items into consideration, and are included in the R.Ae.S. Data Sheets mentioned earlier.

Example 12.1. The following particulars relate to a thin web beam, which may be regarded as of the complete tension field type.

Depth of web panel = depth between flange centroids = 6 in.

Web thickness = 0.022 in.

Distance between stiffeners = 4.5 in.

Cross-sectional area of stiffener = 0.0694 sq. in.

" " " " flange = 0.319 sq. in.

Z_c for flange = 0.068 in.³

Assuming the simple theory to apply, determine the tensile stress in the web, the compressive stress in the stiffener, and the maximum compressive stress in the flange, if, at the section considered, the shear force is 1250 lb., and the bending moment is 18,510 lb. in.

From equation 12.12, the tensile stress in the web is given by

$$f_t = \frac{2S}{dt} = \frac{2 \times 1250}{6 \times 0.022} = 18,940 \text{ lb. per sq. in.}$$

From equation 12.13, the compressive load in the stiffener is

$$V = \frac{Sb}{d} = \frac{1250 \times 4.5}{6.0} = 937.5 \text{ lb.}$$

and the corresponding stiffener stress = $\frac{937.5}{0.0694} = 13,520 \text{ lb. per sq. in.}$

From equation 12.14, the maximum compressive stress in the flange is

$$\begin{aligned} f_c &= \frac{\frac{18,510}{6} + \frac{1250}{2}}{0.319} + \frac{1250 \times (4.5)^2}{12 \times 6 \times 0.068} \\ &= \frac{3085 + 625}{0.319} + 5170 \\ &= 11,630 + 5170 = 16,800 \text{ lb. per sq. in.} \end{aligned}$$

THE INSTABILITY OF THIN PLATES

13.1. Introductory

MODERN stressed skin aircraft construction makes use of the load carrying capacity of the thin metal sheets used for the wing, fuselage and other surface coverings. The thin sheets are often subjected to loads which produce compression or shear in the planes of the sheets. We have mentioned earlier (para. 5.10) that thin plates will buckle into waves at quite low values of a compressive stress, and that, since shear produces a diagonal compression, this phenomenon will occur also for a plate in shear. In certain instances it may be inadvisable to allow the sheet to reach the buckled state, even though failure does not necessarily occur as a result of buckling, so that it is essential that we shall be able to predict the stress at which buckling will commence. Even if we make use of the load carrying capacity of the sheet after buckles have formed, we must know the stress at which the buckling starts in view of the changed stress distribution which arises after buckling. The theoretical solution of the problem of instability of a flat plate has been considered by many investigators, and much experimental work has been, and is still being, done to provide design data. The theory is beyond the scope of this book, and some of the results only will be quoted here. In order to indicate the form of the expression for the theoretical buckling stress for a thin plate in compression, a simplified theory, which is by no means rigorous, will be presented in the next paragraph. The student should realise that this is not to be regarded as a derivation of the buckling stress, since it neglects certain parameters, and involves some simplifying assumptions, which cannot properly be made if a rigorous solution is desired.

13.2. Simplified Theory for a flat plate subjected to edge compression

The flat plate ABCD in Fig. 13.1 is loaded with a compressive load P , uniformly distributed over the edges AD and BC, so that the compressive stress, p , is equal to $\frac{P}{bt}$ where t is the plate thickness. We consider an elementary strip of the plate, EF, of width δy , and assume the plate to be given a small displacement, so that this element bows in the form shown in Fig. 13.2, the deflection at x from AD

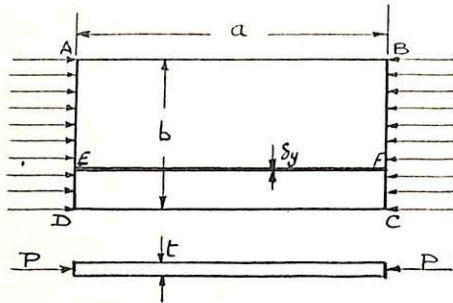


FIG. 13.1.—FLAT PLATE SUBJECTED TO EDGE COMPRESSION.

being w . The compressive load on this elementary strip is $\delta P = p t \delta y$, and the bending moment at x from AD is given by

$$M = -\delta P \cdot w$$

The longitudinal stress due to bending is $f_x = \frac{M \cdot t/2}{\delta I}$, where δI is the second moment of area of the cross-section of the strip. If the strip were completely isolated, the longitudinal strain, f_x/E , would be accompanied by a lateral strain

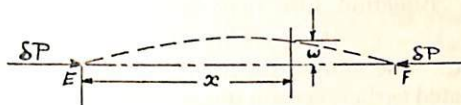


FIG. 13.2.—DEFLECTION OF ELEMENT OF PLATE.

of amount $-\sigma \cdot f_x/E$, where σ is Poisson's ratio. Due to the proximity of other such strips, however, this lateral strain is prevented. (This effect will always arise if the width of the plate is large compared with its thickness.) The prevention of this strain is equivalent to the imposition of a lateral strain of amount $+\sigma f_x/E$, which, in turn, produces a further longitudinal strain of amount $-\sigma^2 f_x/E$. Hence, the total longitudinal strain, ϵ_x , is given by

$$\begin{aligned} \epsilon_x &= \frac{f_x}{E} - \frac{\sigma^2 f_x}{E} \\ &= \frac{f_x}{E} (1 - \sigma^2) \end{aligned} \quad \text{Eq. 13.1}$$

Inserting in equation 13.1 the value of f_x given above, we have

$$\epsilon_x = \frac{M t (1 - \sigma^2)}{2 E \cdot \delta I} \quad \text{Eq. 13.2}$$

If R is the radius of curvature of the neutral surface, assumed to be the mid-surface of the plate, we may show that, following the simple bending assumption that plane cross-sections remain plane,

$$\epsilon_x = \frac{t/2}{R} \quad \text{Eq. 13.3}$$

and further,

$$\frac{1}{R} = \frac{d^2 w}{dx^2} \quad \text{Eq. 13.4}$$

Then from equations 13.2, 13.3, 13.4,

$$\frac{d^2 w}{dx^2} = \frac{M (1 - \sigma^2)}{E \cdot \delta I}$$

and, as $M = -\delta P \cdot w$, this becomes, if we write $E' = \frac{E}{1 - \sigma^2}$,

$$\frac{d^2 w}{dx^2} = - \frac{\delta P}{E' \cdot \delta I} \cdot w \quad \text{Eq. 13.5}$$

Equation 13.5 is of exactly the same form as equation 10.1, if E' is substituted for E in the latter. Hence, the critical value of δP is given by

$$\delta P_{cr} = \frac{n \pi^2 E' \delta I}{a^2}$$

where n is an integer. We note that $\delta I = \frac{\delta y \cdot t^3}{12}$, and that the cross-sectional area of the strip is $\delta A = t \delta y$, so that the critical stress, p_{cr} , is

$$p_{cr} = \frac{\delta P_{cr}}{\delta A} = \frac{n\pi^2 E t^2}{12(1 - \sigma^2)a^2} \quad \text{Eq. 13.6}$$

We assume that a similar expression holds for all such elementary strips, and in particular that an expression of this form gives the critical compressive stress for the plate as a whole. Equation 13.6 may be written in the form,

$$p_{cr} = KE(t/b)^2 \quad \text{Eq. 13.7}$$

where K is a constant. The simple form of K indicated by the above analysis is incorrect, since, as stated earlier, certain items which affect the result are neglected in this simple approach. For example, no account is taken of the fact that the deflection, w , will depend on y , the distance from CD, as well as on x . In fact, in the detailed theory it is usual to assume a deflection curve of the form $w = w_0 \sin\left(\frac{n\pi x}{b}\right) \sin\left(\frac{m\pi y}{a}\right)$. However, equation 13.7 is the form in which the

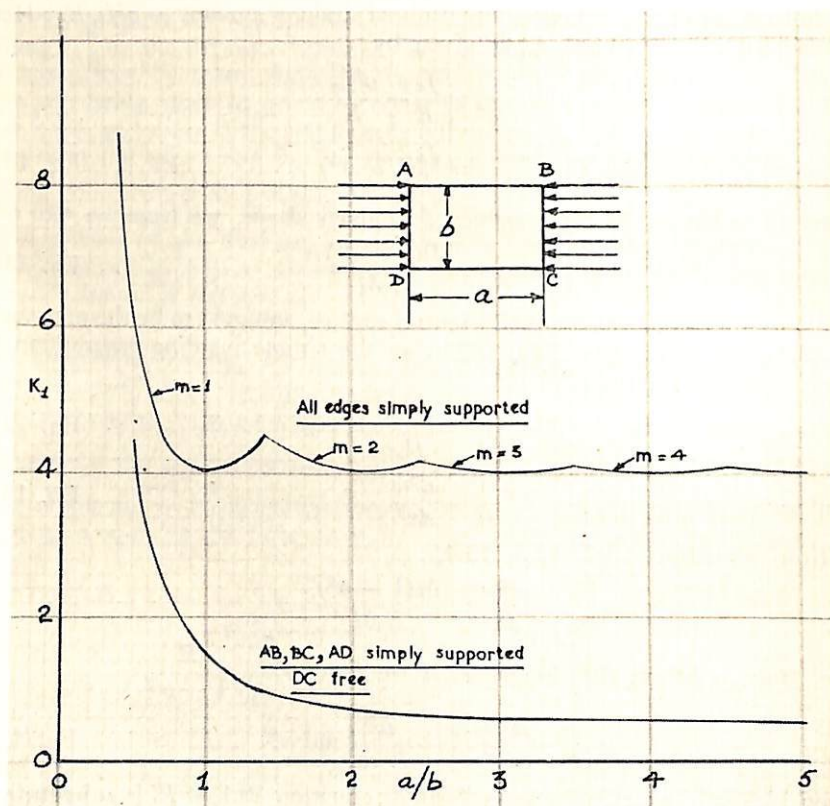


FIG. 13.3.—FLAT PLATES IN COMPRESSION.

$$\text{CRITICAL STRESS, } p_{cr} = K_1 \cdot \frac{\pi^2 E}{12(1 - \sigma^2)} \cdot \left(\frac{t}{b}\right)^2$$

critical stress is usually expressed, the constant K depending on the ratio a/b , the number of half-waves in the sheet, and on the conditions of edge support. G. H. Bryan derived the following expression (*Proceedings of the London Mathematical Society*, Vol. XXII, 1891, p. 54).

$$p_{cr} = \left(\frac{a}{mb} + \frac{mb}{a} \right)^2 \frac{\pi^2 E}{12(1 - \sigma^2)} \cdot \left(\frac{t}{b} \right)^2 \quad \text{Eq. 13.8}$$

where m is the number of half-waves in the length direction, the number of half-waves in the width direction being taken as one, as the analysis shows this to give the smallest value of the critical load. Equation 13.8 refers to the case where all four edges are simply supported.

Writing equation 13.8 in the form

$$p_{cr} = K_1 \cdot \frac{\pi^2 E}{12(1 - \sigma^2)} \cdot \left(\frac{t}{b} \right)^2 \quad \text{Eq. 13.9}$$

we may plot values of K_1 , as shown in Fig. 13.3, against a/b , for different values of m . This equation holds for other conditions of edge support, and Fig. 13.3 shows also the values of K_1 for a plate with three edges simply supported and one edge free.

13.3. Critical Loads for Thin-walled Struts

Several empirical methods exist for predicting the critical loads for struts which are built up of thin metal elements. For preliminary design purposes, perhaps the most commonly used method is the following. A section such as that illustrated in Fig. 13.4 is regarded as being made up of the five distinct elements, numbered from 1 to 5. The critical load for each element is calculated, on the basis of equation 13.9 or from experimental data, and the loads thus found for all elements are added to give the critical load for the strut. If used with discretion,

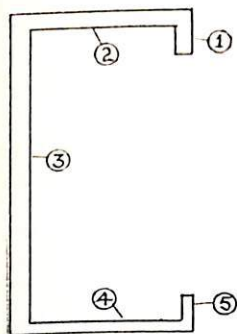


FIG. 13.4.—THIN-WALLED STRUT REGARDED AS MADE UP OF A NUMBER OF RECTANGULAR ELEMENTS.

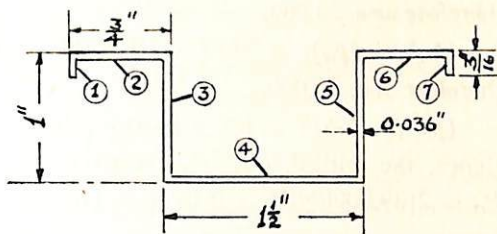


FIG. 13.5.—EXAMPLE 13.1.

this method gives a reasonably close approximation to the compressive strength of such a strut. Certain points should be noted:—

(i) If the critical stress for any element is greater than the yield stress, the latter should be used to determine the critical load for that element. There are exceptions to this rule. For example, the critical stress for elements 1 and 5 in Fig. 13.4, for which the dimension b is very small, may be above the yield stress, but it is

very unlikely that the yield stress will be reached before the other elements buckle, and in such a case the critical load for element 1 should be based on the critical stress for element 2.

(ii) The critical stress for elements such as 1 or 5 is calculated assuming one edge free and three edges simply supported, whilst for elements 2, 3, 4 all four edges are assumed simply supported.

The following numerical example should make the procedure quite clear:—

Example 13.1. Regarding the section shown in Fig. 13.5 as made up of flat plate elements, calculate the critical compressive load for a strut of this section which is 6 in. long. $E = 10^7$ lb. per sq. in., and $\sigma = 0.3$. The yield stress is 39,000 lb. per sq. in.

For elements 1 and 7, a/b is large, one edge is free and three edges are simply supported. From Fig. 13.3, we see that the value of K_1 is 0.5. For elements 2, 3, 4, 5, 6 the ratio a/b is 4 or greater, all four edges are simply supported, and from Fig. 13.3 we derive the value of K_1 as 4.0.

For elements 1 and 7, the critical stress is then

$$p_{cr} = \frac{0.5\pi^2 E}{12(1 - \sigma^2)} \cdot \left(\frac{t}{b}\right)^2 = 0.452E \left(\frac{t}{b}\right)^2$$

and for the remaining elements we have

$$p_{cr} = \frac{4.0\pi^2 E}{12(1 - \sigma^2)} \cdot \left(\frac{t}{b}\right)^2 = 3.617E \left(\frac{t}{b}\right)^2$$

Using centre line dimensions, from the figure,

$$\begin{aligned} b_1 &= 0.1695 \text{ in.} = b_7 & b_2 &= 0.714 \text{ in.} = b_6 \\ b_3 &= 0.964 \text{ in.} = b_5 & b_4 &= 1.464 \text{ in.} \end{aligned}$$

For all elements, $t = 0.036$ in.

$(p_{cr})_1 = (p_{cr})_7 = 0.452 \times 10^7 \times (0.036/0.1695)^2 > 39,000$ lb. per sq. in.
therefore use 39,000.

$(p_{cr})_2 = (p_{cr})_6 = 3.617 \times 10^7 \times (0.036/0.714)^2 > 39,000$ lb. per sq. in.
therefore use 39,000.

$(p_{cr})_3 = (p_{cr})_5 = 3.617 \times 10^7 \times (0.036/0.964)^2 > 39,000$ lb. per sq. in.
therefore use 39,000.

$(p_{cr})_4 = 3.617 \times 10^7 \times (0.036/1.464)^2 = 21,850$ lb. per sq. in.

Hence, the critical load, P_{cr} , is given by

$$\begin{aligned} P_{cr} &= 2\{39,000(0.036 \times 0.1695 + 0.036 \times 0.714 + 0.036 \times 0.964)\} \\ &\quad + 21,850 \times 0.036 \times 1.464 \\ &= 5185 + 1152 = 6337 \text{ lb.} \end{aligned}$$

13.4. Sheet-stiffener Combinations

If stiffeners are attached to a panel which is subjected to compression, the stiffeners, of course, will assist in carrying the load, acting as struts. Up to the point at which the sheet buckles, it is reasonable to assume that the compressive stress in the sheet is uniformly distributed between the stiffeners, and that, if the sheet and stiffener materials have the same stress-strain characteristics, the load will be shared between the sheet and the stiffeners in proportion to their cross-sectional areas. When the sheet has buckled, however, this is no longer

the case. The stress in the sheet varies across the width, being greater near the stiffeners than it is midway between them. In fact, if the compressive load can be sufficiently increased without absolute failure of the stiffened panel, the stress in the centre of the panel may even become tensile.

Final failure of the stiffened panel will usually appear as "strut failure" of the stiffeners. It is not, however, a simple problem to determine the stress at which such failure occurs, since part of the sheet undoubtedly works with the stiffeners, modifying the effective stiffener cross-section. The amount of sheet working with a stiffener varies with the distribution of stress over the width of the panel, and the determination of the failing load usually involves a series of approximations. The most readily available data for the design of stiffened panels in this country is contained in the Royal Aeronautical Society's Data Sheets on "Stressed Skin Construction." These data sheets comprise curves embodying the results of recent theoretical and experimental work, and contain references to the sources from which more detailed information may be obtained.

For rough initial design purposes, an approximation often made is to assume an "effective width" of sheet of about thirty times the thickness to be working with a stiffener, and to base the failing load on the strut failure of the modified stiffener section thus obtained. This value of $30t$ applies for an intermediate stiffener, a length of $15t$ being taken on either side of the stiffener.

13.5. Flat Plates in Shear

Since shear gives rise to diagonal compression, we should expect an expression for the critical shear stress, at which buckling commences, to be of the same form as that for the critical compressive stress. This is the case, and the buckling shear stress is given by

$$\tau_{cr} = K_s \frac{\pi^2 E}{12(1 - \sigma^2)} \left(\frac{t}{b} \right)^2 \quad \dots \quad \text{Eq. 13.10}$$

where b is the shorter dimension of the rectangular sheet.

The theoretical values of K_s are plotted in Fig. 13.6. Experimental work indicates generally that the theoretical values of K_s are somewhat too high, and a number of empirical formulæ have been suggested.

Although the formulæ have been derived usually for rectangular panels, the critical stress for a panel of varying depth may be estimated reasonably by using the mean dimensions.

It should be remembered that shear buckling of a panel does not necessarily constitute failure, and that it is possible for a buckled panel to continue to carry load. The buckling will, of course, induce bending stresses, and until the stress at some point exceeds the elastic limit, the buckles will disappear on removal of the load.

13.6. Effect of Buckling on Shear Stiffness

If a thin plate is subjected to a shear stress of such a magnitude that it buckles, forming a diagonal tension field, the ratio of shear stress to shear strain will not be the same as for the unbuckled plate, even though the elastic limit is not exceeded. The effect of buckling is to reduce the shear stiffness, and this reduction can be represented by the use of an "effective modulus of rigidity," G_e , in the shear stress-strain relationship. For a complete tension field, in which the whole of the shear

is carried by diagonal tension, the value of G_c may be determined simply as follows, for small strains such as will exist if the elastic limit is not to be exceeded.

In Fig. 13.7, AC represents a tension diagonal in the plate under shear. If AC moves to AC' as a result of the distortion of the plate, the extension in AC may

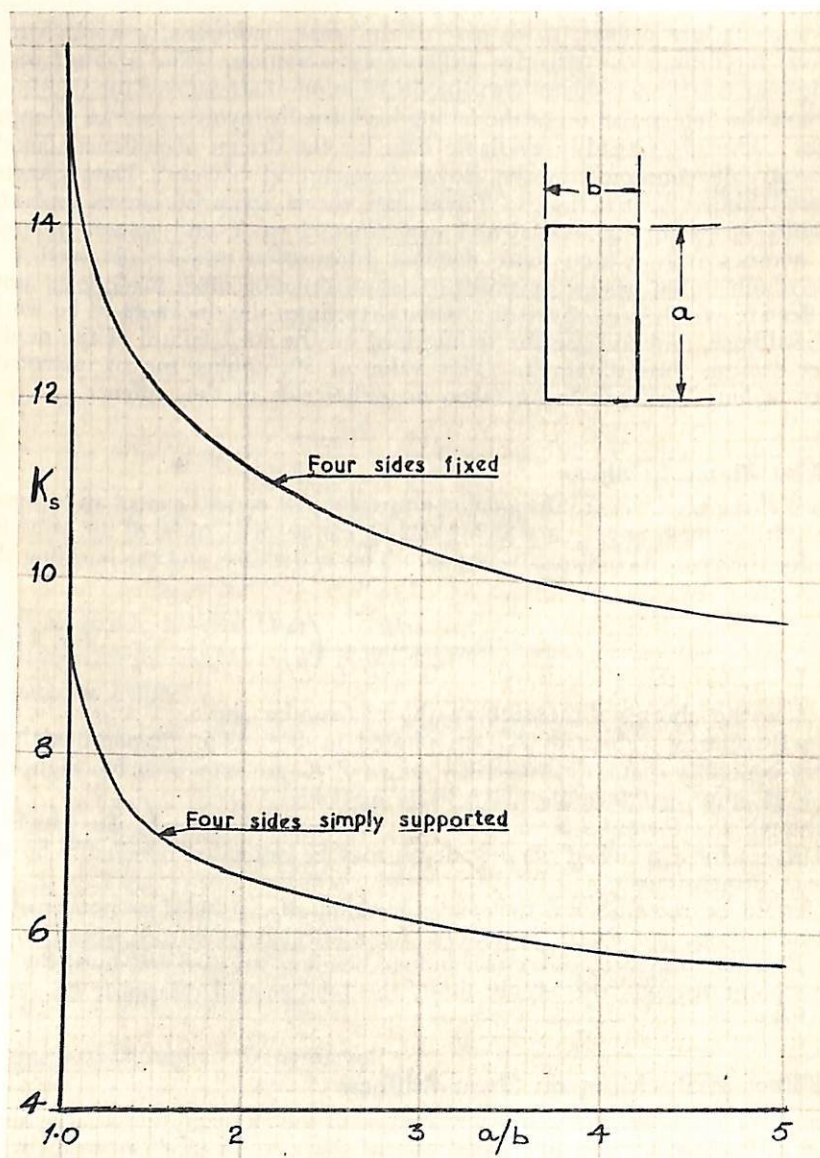


FIG. 13.6.—FLAT PLATES IN SHEAR.

$$\text{CRITICAL SHEAR STRESS, } q_{cr} = K_s \frac{\pi^2 E}{12(1 - \sigma^2)} \left(\frac{t}{b}\right)^2.$$

be approximated to by the length DC' , where CD is perpendicular to AC' . Then the tensile strain in $AC = \epsilon_t = \frac{DC'}{AC} = \frac{CC' \cos CC'D}{AC} = \frac{\delta \times \cos CC'D}{AC}$

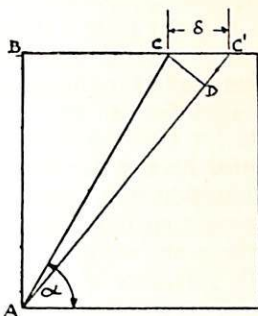


FIG. 13.7.—EXTENSION OF DIAGONAL IN A PLATE IN SHEAR.

For small strains, the angle $CC'D$ is approximately equal to α .

Hence,
$$\epsilon_t = \frac{\delta \times \cos \alpha}{AC} = \frac{\delta \times \cos \alpha}{AB/\sin \alpha} = \frac{\delta}{AB} \times \sin \alpha \cos \alpha$$

But
$$\frac{\delta}{AB} = \text{shear strain} = \phi$$

so that
$$\phi = \frac{\epsilon_t}{\sin \alpha \cos \alpha} = \frac{2\epsilon_t}{\sin 2\alpha}$$

From equation 12.4, we know that the diagonal tensile stress, f_t , is given by $f_t = \frac{2q}{\sin 2\alpha}$, and, since the elastic limit is not exceeded,

$$f_t = E \times \epsilon_t$$

Writing $G_e = q/\phi$, we have, therefore,

$$G_e = \frac{q \sin 2\alpha}{2\epsilon_t} = \frac{q \sin 2\alpha}{2} \times \frac{E}{f_t} = \frac{E \sin^2 2\alpha}{4}$$

But, from equation 3.21, we know that $G = \frac{E}{2(1+\sigma)}$, which, taking σ as 0.25, gives $G = \frac{2}{5}E$.

Hence
$$\frac{G_e}{G} = \frac{5}{8} \times \sin^2 2\alpha$$

In the complete diagonal tension field theory, it is assumed that α is 45° , so that we take $\sin 2\alpha$ to be 1.

Then

$$\frac{G_e}{G} = \frac{5}{8}$$

or

$$G_e = \frac{5}{8}G$$

It should be noted that this relationship is true only for a complete tension field, with α equal to 45° , in which the elastic limit is not exceeded, and for a material for which Poisson's ratio is 0.25. Due account should be taken of this reduction in shear stiffness when calculating the angle of twist of a thin-walled shell, and also when applying equation 6.8 in the determination of the shear distribution in the walls of a torsion tube.

STRUCTURAL DESIGN REQUIREMENTS

14.1. In this country the design loads for the various parts of the structure of an aeroplane are determined according to the requirements specified by two authorities. The requirements for civil aircraft are contained in "Civil Airworthiness Requirements," issued by the Air Registration Board, and those for military aeroplanes in Air Publication 970, issued by the Air Ministry.

This chapter will contain merely an indication of the general nature of these requirements, the details of which are modified from time to time in the light of experience. For full details reference should be made to the publications mentioned above.

14.2. The Flight Envelope

The loads in flight to which an aeroplane structure is to be designed are based on various combinations of speed and load factor specified by a *flight envelope*, which is a diagram of the general form shown in Fig. 14.1. The shape of the actual flight envelope to be used depends on the type of aeroplane and the purpose

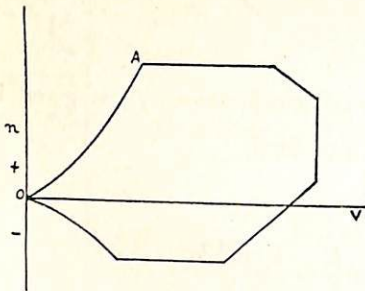


FIG. 14.1.—GENERAL FORM OF FLIGHT ENVELOPE.

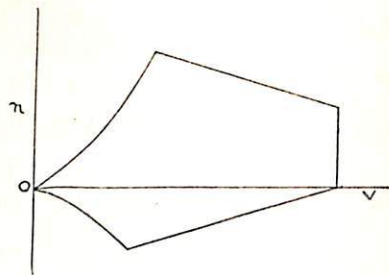


FIG. 14.2.—POSSIBLE FORM OF FLIGHT ENVELOPE FOR AN ULTRA-LIGHT AEROPLANE.

for which it is designed. As an illustration of the extent to which the flight envelope may vary from the general form of Fig. 14.1, the form for ultra-light aeroplanes, as given in the A.R.B. Draft Requirements for ultra-light aircraft, is shown in Fig. 14.2.

The requirements state that the aircraft structure must be designed for any combination of load factor (n) and speed (V) within or on the boundary of the appropriate flight envelope. Remembering that we defined load factor in I.7.4 as the ratio of the lift in a given flight condition to the lift in steady horizontal flight at the same speed, (i.e., generally $n = L/W$), we may determine the curved portion of the flight envelope. At any given speed the maximum load factor attainable is obviously reached if the angle of attack is that at which the maximum lift coefficient ($C_{l_{\max}}$) is obtained. This is the condition at all points on the curve OA. It will be apparent that theoretically this curve may be extended beyond A, but practical limitations are imposed on the maximum load factor to which an aeroplane is required to be designed. These limitations may be physiological—the acceleration which the human body can withstand is limited, the

amount varying to quite an appreciable extent with position, as experiments have shown—and they may also be based on past experience with aircraft of particular types. Even if the physiological limitations did not exist, it would not be possible to design a modern aeroplane for the maximum possible load factor, as the structure weight would become excessive.

We have mentioned acceleration above, because the specification of a load factor n really implies the existence of an acceleration normal to the flight path of $(n - 1)g$, where g is the acceleration due to gravity. This will be explained more fully in the next section.

Reverting to the portion OA of the flight envelope, let V_s be the stalling speed, which is the steady horizontal flight speed corresponding to $C_{L_{\max}}$. Then we know that in the case of flight at this speed the lift, L , is equal to $C_{L_{\max}} \times Sq_s$, where S is the wing area and q_s is $\frac{1}{2}\rho V_s^2$ (1.6.2). At any other speed, V , the lift corresponding to $C_{L_{\max}}$ is equal to $C_{L_{\max}} \times Sq$, where q is $\frac{1}{2}\rho V^2$. Consequently, the load factor experienced when the angle of attack at speed V is such as to give $C_{L_{\max}}$ is given by

$$n = \frac{L}{W} = \frac{C_{L_{\max}} Sq}{C_{L_{\max}} Sq_s} = \frac{V^2}{V_s^2} \quad \text{Eq. 14.1}$$

It should be noted that V_s is based on the value of $C_{L_{\max}}$ for the complete aeroplane, which will generally be different from that for the wing alone. For an aeroplane whose stalling speed is 100 miles per hour and whose top speed is 400 miles per hour, the maximum possible load factor would be 16. Quite obviously, for a case like this, the structure would have to be very heavy, since the maximum loads would be 16 times as big as those experienced in steady flight.

14.3. Determination of Loads in Accelerated Flight

Considering an aeroplane flying in a manoeuvre such that the total lift is n times the weight, the flight path being assumed instantaneously horizontal, there is an unbalanced force normal to the flight path of amount $(L - W)$, or $(n - 1)W$. A body of mass m subjected to an unbalanced force, F , experiences an acceleration in the direction of the applied force of amount a , given by $F = ma$. Consequently, the aeroplane considered experiences an acceleration of amount $\frac{(n - 1)W}{\text{mass}}$ in a

direction normal to the flight path. Writing the mass as W/g , we can express this normal acceleration as $(n - 1)g$. In order to reduce a problem involving accelerations into one which can be solved by methods of statics, we introduce the concept of *inertia forces*. We know the magnitude of the *unbalanced* force required to produce a given acceleration, a , on a body of mass m as equal to the product ma . Consequently, a force of this magnitude applied in a direction opposite to that of the acceleration must produce equilibrium. Such a *reversed effective* force is commonly termed an *inertia force*. Applying, then, a force of magnitude $(n - 1)W$ to our aeroplane, in a direction opposing the lift, has the effect of balancing the forces normal to the flight path, and we have equilibrium as far as this particular direction is concerned. Since an increase in lift will generally be accompanied by an increase in drag, it is necessary to introduce also a *forward inertia force*, F . It must be emphasised that the lift, referred to above as L , is correctly the total aerodynamic lift, including the tail lift as well as the

wing lift. Angular acceleration in pitch is neglected in some cases, but it must be taken into account in other specified cases and it must be balanced by the introduction of appropriate inertia forces and couples.

Having introduced the inertia forces, the system of forces to be considered now becomes that shown in Fig. 14.3, and the unknown quantities are determined

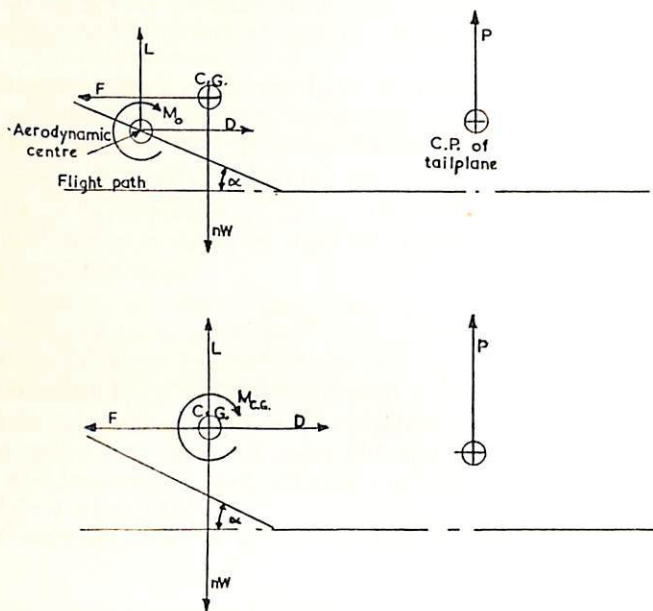


FIG. 14.3.—ALTERNATIVE REPRESENTATIONS OF THE LOADS ON AN AEROPLANE IN ACCELERATED FLIGHT.

from the usual conditions of static equilibrium. In certain instances, when the angle of attack is not specified, it may be necessary to proceed by a method of successive approximations. Numerical examples follow to illustrate possible methods of performing the *balance-out calculations* for different combinations of load factor and speed. In the cases considered in these examples, we assume "engine-off."

Example 14.1. The curves of C_D , α and $C_{M_{C.G.}}$ for a light aeroplane are given in Fig. 14.4, and relevant dimensions are shown in Fig. 14.5. The following data are also given.

$$\begin{aligned}\text{Weight} &= 1800 \text{ lb.} \\ \text{Wing area} &= S = 159 \text{ sq. ft.} \\ \text{Mean chord} &= 4.49 \text{ ft.}\end{aligned}$$

Determine the lift, drag, tail load and forward inertia force for the two following cases:—(i) $n = 4.5$, aircraft in stalled attitude, (ii) $n = 4.5$, $V = 141$ m.p.h.

(i) Since we know the aircraft is in the stalled attitude, we know the angle of attack, the value of C_L , C_D and $C_{M_{C.G.}}$ from the curves. We see that $\alpha = 19^\circ$, $C_L = 1.47$, $C_D = 0.153$, $C_{M_{C.G.}} = 0.112$.

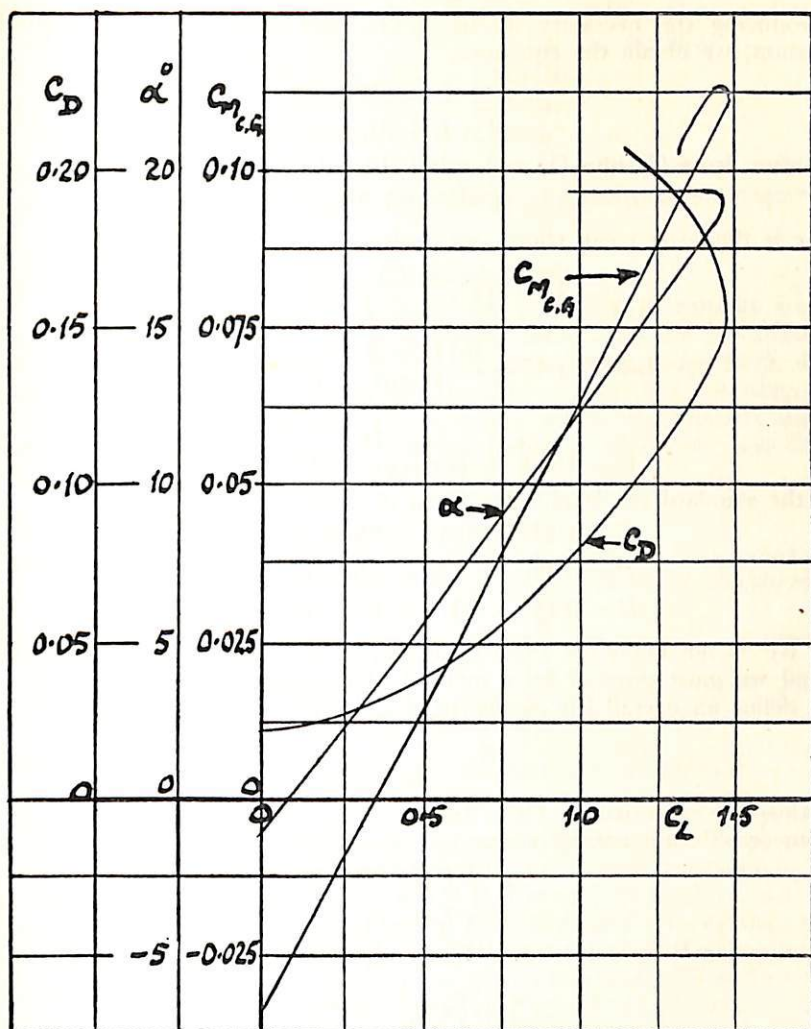


FIG. 14.4.

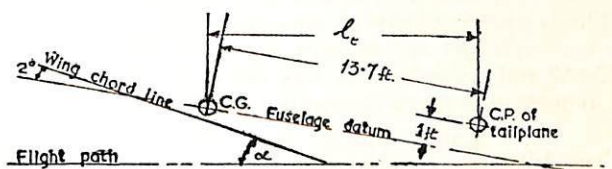


FIG. 14.5.—DIMENSIONS FOR EXAMPLE 14.1.

From the geometry of Fig. 14.5, we calculate the length of the tail arm, measured parallel to the flight path, as

$$\begin{aligned}
 l_t &= 13.7 \cos(\alpha - 2^\circ) + 1.0 \sin(\alpha - 2^\circ) \\
 &= 13.1 + 0.293 = 13.393 \text{ ft.}
 \end{aligned}$$

Introducing the necessary inertia forces, and applying the conditions of equilibrium, we obtain the equations,

$$L + P = nW \quad \dots \dots \dots (1)$$

$$P \times l_t = M_{C.G.} \quad \dots \dots \dots (2)$$

Substituting from (2) into (1) and using the relationships

$$L = C_L \times \frac{1}{2}\rho SV^2 \text{ and } M_{C.G.} = C_{M_{C.G.}} \times \frac{1}{2}\rho SV^2 c,$$

where c is the wing mean chord, we derive

$$\frac{1}{2}\rho SV^2 \left(C_L + \frac{C_{M_{C.G.}} c}{l_t} \right) = nW$$

$$\frac{1}{2}\rho SV^2 \left(1.47 + \frac{0.112 \times 4.49}{13.393} \right) = 4.5 \times 1800$$

$$\frac{1}{2}\rho SV^2 = 5375 \text{ lb.}$$

Then

$$V = 168.5 \text{ ft. per sec.} = 115 \text{ m.p.h.}$$

(using the standard sea level value for ρ of 0.00238 slugs per cu. ft.).

$$L = 1.47 \times 5375 = 7900 \text{ lb.}$$

$$P = 8100 - 7900 = 200 \text{ lb.}$$

$$D = 0.153 \times 5375 = 823 \text{ lb.} = F.$$

(ii) We do not know the angle of attack, so that we cannot evaluate the tail arm, and we must proceed by a method of continuous approximation.

We define an overall lift coefficient, $\bar{C}_L$, by the equation

$$\bar{C}_L = \frac{nW}{\frac{1}{2}\rho SV^2} \quad \dots \dots \dots \text{Eq. 14.2}$$

In this case we calculate $\bar{C}_L = 1.0$.

From equilibrium considerations,

$$L + P = nW \quad \dots \dots \dots (1)$$

and

$$P \times l_t = M_{C.G.} \quad \dots \dots \dots (2)$$

Substituting for P from (2) into (1), and dividing through by $\frac{1}{2}\rho SV^2$,

$$C_L + \frac{c}{l_t} C_{M_{C.G.}} = \bar{C}_L \quad \dots \dots \dots (3)$$

As a first approximation, assume that α is that corresponding to a value of C_L equal to $\bar{C}_L$. (This is equivalent to neglecting the tail load in a first approximation.) Then, from the curves, a first approximation to α is 12.5° . Using this value we calculate $l_t = 13.662$ and we substitute this value, and the value of $C_{M_{C.G.}}$ from the curves (i.e., 0.0625) into (3) to determine our second approximation to C_L .

$$\text{Thus, } C_L = 1.0 - \frac{4.49 \times 0.0625}{13.662} = 1.0 - 0.0205 = 0.9795$$

Using this value as a second approximation, we obtain $\alpha = 12^\circ$, giving $l_t = 13.678$ ft., and $C_{M_{C.G.}} = 0.059$, and from (3) the third approximation follows as $C_L = 1.0 - 0.0194 = 0.9806$. It is obvious that any attempt at a further approximation will not give a value of C_L differing very much from that just obtained, and we take this value, therefore, as the solution.

Hence, $C_L = 0.9806$, and from the curves $C_D = 0.078$, $C_{M.c.g.} = 0.059$ and since $\frac{1}{2}\rho SV^2 = \frac{1}{2} \times 0.00238 \times 159 \times (207)^2 = 8100$ lb., we have

$$L = 0.9806 \times 8100 = 7940 \text{ lb.}$$

$$P = 8100 - 7940 = 160 \text{ lb.}$$

$$D = 0.078 \times 8100 = 632 \text{ lb.} = F.$$

A similar method of approximation may be adopted in all cases in which the angle of attack is not given, and the number of approximations required will generally be found to be small.

14.4. Limit Loads, Proof Loads, Ultimate Loads

The loads which are determined as in the preceding paragraphs are called *unit loads* or *limit loads*. For design purposes these loads are multiplied by a specified *ultimate factor* (usually 1.5) to yield the *ultimate design loads*, and the structure must withstand without failure the loads so obtained. Another factor, the *proof factor*, specifies the *proof loads*, under which the structure must not suffer detrimental deformation. The proof factor is, of course, less than the ultimate factor.

14.5. Gust Cases

If an aeroplane in horizontal flight encounters a sudden gust of air, which has a vertical component of velocity, there is an instantaneous change in angle

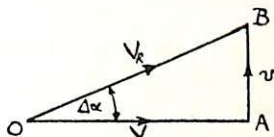


FIG. 14.6.—CHANGE IN ANGLE OF ATTACK DUE TO AN UP-GUST.

of attack with a resulting change in lift. This effect must be allowed for in design, and the aeroplane structure must withstand the load factors due to up and down gusts of prescribed values. The gust velocities to be used are based on observations made in various parts of the world and at various heights. Although it is possible for an aeroplane to encounter a gust of greater velocity than that for which it has been designed, the frequency of occurrence of the higher speed gusts is small, and it is not considered necessary to design for the maximum gust velocity that has been recorded. It may well be, however, that as aeroplanes fly at increasing speeds at greater heights, the prescribed values of gust velocity will be increased.

The effect of a gust with an upward component of velocity, v ft. per sec., on an aeroplane flying at a speed of V ft. per sec. may be seen by reference to Fig. 14.6.

In this figure, OA represents the *relative* air velocity, V , and AB the gust velocity, v , so that OAB is a triangle of velocities, OB being the resultant velocity of the air relative to the aeroplane. The instantaneous change in angle of attack, $\Delta\alpha$, is the angle BOA, so that

$$\Delta\alpha = \tan^{-1} \frac{v}{V}$$

Since v is assumed to be small compared with V , we may write

$$\Delta\alpha = \frac{v}{V} \text{ radians}$$

and approximately,

$$V_R = V$$

Consequently, if the slope of the curve of C_L plotted against α is known, its value being $\frac{dC_L}{d\alpha}$, we obtain the instantaneous change in lift coefficient as

$$\Delta C_L = \Delta \alpha \times \frac{dC_L}{d\alpha}$$

Then the instantaneous increase in lift is given by

$$\Delta L = \Delta C_L \times \frac{1}{2} \rho S V^2$$

and the load factor, n , is

$$n = \frac{W + \Delta L}{W} = 1 + \frac{\Delta L}{W}$$

the aeroplane being initially in steady horizontal flight with the lift equal to the weight.

The above assumes that the gust is "sharp-edged"—i.e., that it reaches its maximum velocity instantaneously. In fact this is not so. The gust velocity builds up gradually to its maximum value, although this building up may take place over a very short distance. In order to take account of the relief afforded by this gradual building up of gust velocity, an *alleviating factor*, F , is introduced, and we write

$$\Delta \alpha = F \times \frac{v}{V}$$

The value of F varies according to the type of aeroplane.

14.6. Asymmetric Manœuvres

The cases mentioned in the preceding paragraphs all refer to symmetric manœuvres, in which flight occurs in the longitudinal plane of symmetry of the aeroplane. There are also requirements covering asymmetric manœuvres, which give additional stressing cases, particularly for the tail unit. In the simplest form such a stressing case might require merely an asymmetric distribution of the tail load required for balance in the symmetric case, but in general it is necessary to cater for a specific angular acceleration of the aeroplane about one of its axes.

14.7. Landing Loads

The undercarriage, and the aircraft structure generally, must be designed to withstand the loads which arise in landing under various conditions. The number of separate cases to be considered is generally large, and it is not possible to go into detail here. The undercarriage and aircraft structure must be stressed for landing with a specified velocity of descent, with the aircraft in a specified attitude, and under various conditions of drag and side loads. There are also cases covering taxiing on the ground.

14.8. Other Requirements

There are a number of special requirements for particular components of the structure. For instance, engine mountings must be designed to carry torque and thrust, as well as the inertia loads arising from the flight cases. The control surfaces and the control system have their own set of requirements. The danger of flutter introduces regulations prescribing the stiffness which must be provided in the various component parts, and the degree of balancing of the control surfaces.

The problem of flutter, which involves aerodynamic, elastic, and inertia forces, will be considered in another volume of this book. It suffices, at present, to point out that it is very necessary to consider the stiffness of the structure as well as its strength, in order to avoid, as far as is possible, the danger of flutter.

14.9. Wing Stressing

It is impossible to give more than a brief survey of some of the problems involved in the structural analysis of a wing, and only some of the fundamental aspects of wing structural design will be discussed in an elementary manner.

(a) LOADING, SHEAR, BENDING MOMENT AND TORSION

The determination of the spanwise distribution of the aerodynamic loading on a wing is a problem in aerodynamics, and will not be discussed here. We shall assume that the spanwise load distribution is known. It is obviously more convenient to consider the components of the loading normal and parallel to the wing chord, rather than the lift and drag, which are normal and parallel respectively to the line of flight. Assuming, then, that the distribution of the *normal* aerodynamic load along the span is known, it is possible, by graphical integration, to determine the shear force and bending moment diagrams for the wing due to the aerodynamic loading. It must be realised, however, that the inertia force applied to the aeroplane for balance (para. 14.3), implies that all masses in the aeroplane are subjected to the same inertia factors, so that account must be taken of the weight of the wing and any items located therein in determining the resultant shear force and bending moment diagrams. In general, the inertia forces will oppose the aerodynamic forces, so that the shears and bending moments will be reduced.

It is necessary to obtain a diagram showing the torsion on the wing at all points along the span. For stress analysis purposes it would be desirable to know the torque about the shear centre at any particular section, but the location of the shear centre will not be known until the structure has been determined. Consequently, it is usual to evaluate the torque about some axis which is independent of the actual structure, as, for example, the axis which passes through the leading edge of the root section of the wing and which is perpendicular to the aircraft centre line. Torsion due to inertia forces must be included.

When the spanwise distribution of shear, bending moment, and torsion is known, the wing may be analysed by the methods suggested in Chapter VII. Complete detailed analysis of a stressed skin wing requires the use of more refined methods than the simplified ones outlined in Chapter VII, particularly if the bending material is distributed over the skin, instead of being concentrated in the spar flanges. With high-speed modern aircraft, the irregularities caused by the development of tension field waves in the skin panels are obviously undesirable, and there are grounds for increasing the skin thickness, and utilising the skin itself as bending material. The development of sandwich-type construction, in which a low-density shear-carrying material is sandwiched between two sheets of thin metal, is of decided value in this case. Research and development in connection with sandwich construction are still proceeding, and much valuable information is already available, mostly in the form of technical reports.

(b) RIBS

Wing ribs serve two primary purposes :—(i) to transmit the air loads from the skin to the spars, and (ii) to assist in maintaining the wing contour. In order to

present a preliminary idea of rib stressing, a brief discussion of the problem of determining the shear force and bending moment diagrams for ribs follows.

The total load at any rib station is obtained from the spanwise loading diagram for the wing, as indicated in Fig. 14.7. In this diagram the load on rib r is given by the shaded area, and the load at other rib stations is determined similarly. The chord-wise distribution of air load is that appropriate to the angle of attack, and follows from aerodynamic data for the wing. Note that for rib stressing it is not usually considered necessary to include inertia loads, except in cases where concentrated items are attached to the ribs, producing large inertia loads.

Using the simplified analysis of para. 7.3, we apply reactions at the spars such that, if the total load on the rib is P , the reaction at the front spar is $P \times I_F / (I_F + I_R)$ and that at the rear spar is $P \times I_R / (I_F + I_R)$. The resultant of these reactions passes through the centroid of inertia of the two spars, which will not generally coincide with the centre of pressure of the air load on the rib. Consequently, there is an unbalanced couple, which is reacted by torsional shear round the

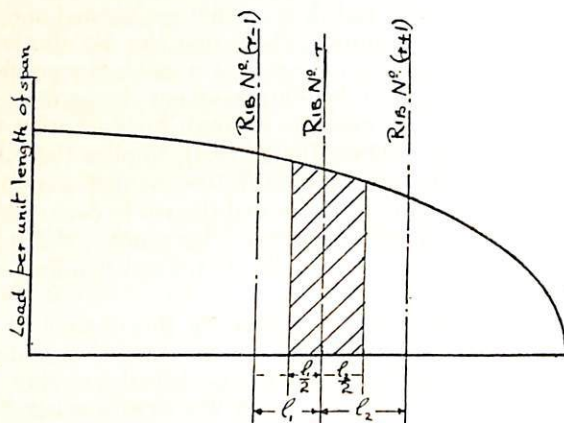


FIG. 14.7.—SPANWISE LOADING DIAGRAM. DETERMINATION OF RIB LOADS.

torsion cells. Having determined the shear flow due to this torsion, it is possible to obtain resultant shear force and bending moment diagrams for the rib by compounding the shear force and bending moment due to the distributed loading and the spar reactions with those quantities due to the torsional shear flow.

(c) NON-STRESSED SKIN WINGS

It has been assumed in the foregoing that the wing covering assists in carrying the loads. If the covering is fabric, its strength will be negligible from the point of view of presenting resistance to torsion, and the torque on the wing must be met entirely by differential bending of the spars. At any chordwise section of the wing, the *shear centre* is now the centroid of the second moments of area of the spars. A load applied at this point will be shared between the spars in proportion to their second moments of area, and reference to Chapter VIII indicates that in such circumstances the spars would deflect equal amounts, so that there will be bending of the wing but no twisting. A torque applied to the wing will produce, assuming a two-spar wing, equal and opposite loads on the two spars, of magnitude equal to the applied torque divided by the distance between the spars. The net effect is that a load applied at any point along the chord will

be taken by the two spars in the inverse ratio of their distances from the point of application of the load.

The *ribs* in such a wing will be regarded as beams supported at the spars, and the spar reactions will be computed in the normal way.

(d) TRUSS-TYPE RIBS

In section (b) above, the determination of shear force and bending moment diagrams for ribs was discussed. It may not always be possible to analyse a rib as a beam in the usual way, making use of the theory of bending to assess the stresses. In the case of a truss-type rib, it is necessary to apply the methods for finding the loads in the members of a frame. In such cases the loading to be applied to the various joints of the frame will be obtained from the chordwise loading diagram in exactly the same way as the rib loads are obtained from the spanwise loading diagram for the wing. There may be occasions when the applied loading produces bending in individual members of the truss, and due account must be taken of such effects.

(e) WING SPARS

In a rigorous stress analysis the wing will be treated as a whole, as mentioned in para. 7.4, but in our simplified preliminary method, we treat the spars separately. The spar reactions determined for the ribs, as described in the previous paragraph, are, of course, now reversed in direction and regarded as concentrated loads applied to the spars by the ribs. A spar, then, becomes a beam with a series of concentrated loads applied to it, and it is a simple matter to draw the corresponding shear force and bending moment diagrams. To these diagrams corrections have to be made, in order to take into account (i) inertia loads, and (ii) the torsional shear in the spars. The latter is simply added to the shear force diagram due to the rib loads and inertia loads, and does not affect the bending moment diagram. Having derived the final shear force and bending moment diagrams, the stresses in the spar may be found by the methods given in earlier chapters.

The problem which presents most difficulty in the design of wing spars is that of joints in the spars. A wing is usually built up in sections, and at the joints between sections provision must be made for transference of shear, bending and torsion. If the bending material is distributed over the surface instead of being concentrated at the spars, and if it is not possible to provide effective joints for all the elements of bending material, means must be provided for transference of load from the non-jointed elements to those which can carry the loads through. Consequently, near a joint, some of the elements will be ineffective in such a case. If joints are provided only in the spars, it is a common practice to make joints merely by means of fittings attached to the flanges, as indicated in the sketch of Fig. 14.8. This type of joint requires that the shear shall be transferred from the web to the flanges and thence to the fittings, necessitating generally a doubling plate on the web in order to avoid undue concentration of stress. Furthermore, since the joint is effected by means of pins, no couple can be transmitted from one fitting into the other, so that any one fitting is loaded as indicated in Fig. 14.9. There will be, therefore, a fixing moment applied to the fitting at its points of attachment to the flange. Alternatively, we may say that the fitting itself applies to the flange an end load, P , a "vertical" shear force, F , and a moment equal to F times the distance from the pin to the centroid of the rivet or bolt group attaching the fitting to the flange. Reinforcement must be provided generally

to cater for this couple and to transform the couples on the two flanges into an end load in each flange. This may be done by further doubling of the web. If a direct joint can be made in the shear webs, this problem will not arise, as

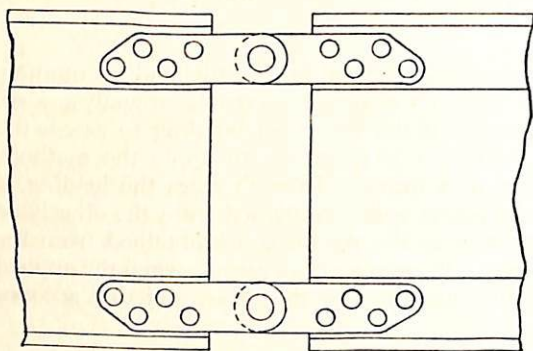


FIG. 14.8.—WING SPAR JOINT.

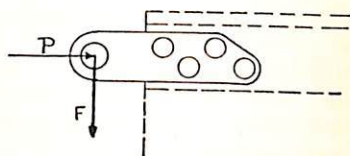


FIG. 14.9.—LOADING ON ONE FITTING OF SPAR JOINT.

the fittings will then be required to transmit only end loads. The following discussion will serve as an illustration of one method of determining the various loads in a wing spar attachment of the type described above.

At station AB, Fig. 14.10, the shear force is S and the bending moment is M_A . The end load to be transmitted by each of the pins A and B is then $P_A = M_A/h$, where h is the depth between the pin centres. We shall assume in this instance that each pin carries one half of the shear force, S , so that the vertical load on each pin is $S/2$. Let the distance from the pin to the centroid of the bolt group

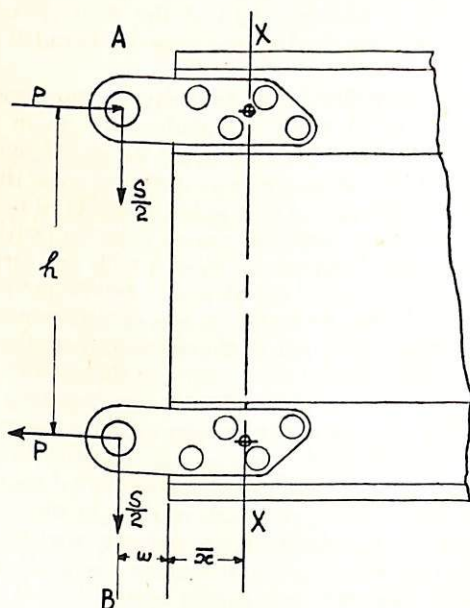


FIG. 14.10.—ANALYSIS OF SPAR JOINT.

attaching the fitting to the flange be $(w + \bar{x})$, as indicated in the diagram. Then the fitting transmits to the flange the following loads:—

- (i) an end load, $P_A = M_A/h$
- (ii) a vertical load, $S/2$
- (iii) a couple of amount $\frac{S}{2}(w + \bar{x}) = \Delta M$.

We may note that the overall moment applied to the spar at XX is

$$P_A \times h - S(w + \bar{x}),$$

which, if we neglect any change in shear between XX and AB, will be the bending moment at station XX. The bending moment in the spar at XX (M_X) gives end loads in the flanges of amount M_X/h_F , where h_F is the depth between flange centroids at XX. For simplicity in our discussion we shall assume that h_F is equal to h . Let the end loads in the flanges due to M_X be P_X . Then assuming that there is ample provision for transference of shear from the web into the flanges and fittings, the loading on the fittings as at present ascertained is as shown in Fig. 14.11 (a). In this diagram, the magnitude of ΔM is as defined above. It will be observed that there is an unbalanced end load of amount $(P_A - P_X)$. If a member attached to both flanges and fittings is inserted as indicated by the dotted lines in the diagram, we may regard this member as a beam, loaded by a couple ΔM at each end by the two fittings, and supported

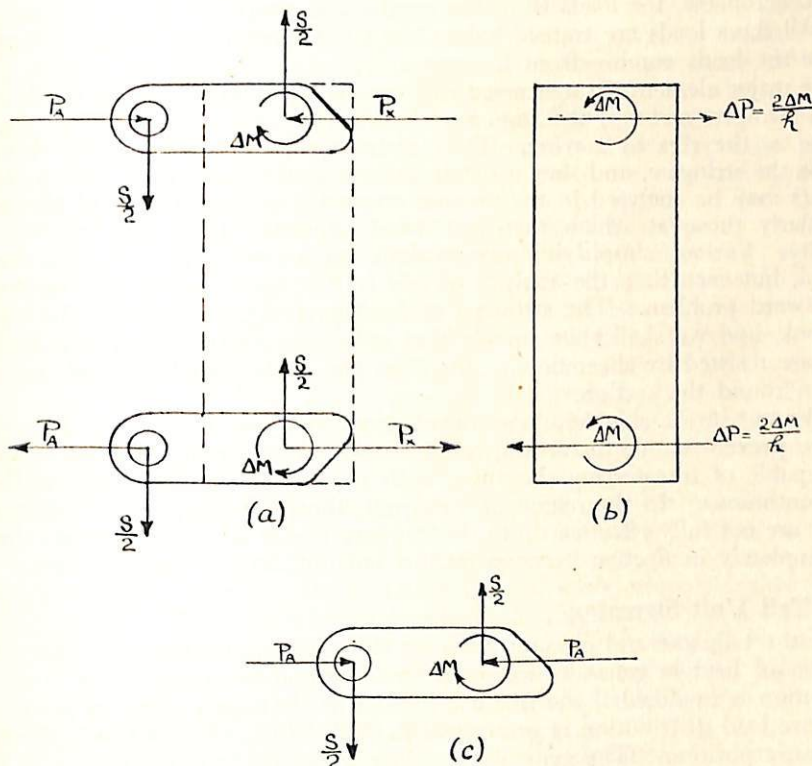


FIG. 14.11.—ANALYSIS OF ELEMENTS OF SPAR JOINT.

by the flanges, so that it is loaded as shown in Fig. 14.11 (b). (Note that the direction of ΔM in this diagram is reversed from that in the previous diagram, since we are now considering loads applied by the fitting.) Thus, the total end load applied to the flange by this member and the fitting together is

$$P = P_A - \Delta P = P_A - 2\Delta M/h = P_X$$

Alternatively, we may say that the total end load reaction to the fitting is equal to

$$P_X + \Delta P = P_A$$

The final loadings are, therefore, as in Fig. 14.11 (b) for the reinforcing member, and Fig. 14.11 (c) for the fitting.

Many different types of wing attachment have been devised, and in general each must be treated as a new problem. The above discussion should indicate means of dealing with certain types, although it is not suggested that this is any more than just one way of considering the problems which arise when it is not possible to carry the shear web through.

14.10. Fuselage Stressing

The loading on a fuselage is comparatively simple, but the detailed stress analysis of a modern stressed skin fuselage presents some very complex problems. The major loads to be considered are those on the tail unit, which produce shear, bending and torsion of the fuselage. In addition, there are localised loads due to the weight of items of equipment, baggage, crew, and passengers. In a single-engined aeroplane, the loads from the engine and propeller must be considered, too. All these loads are transferred to the wing attachment points, where they balance the loads coming from the wing.

The major elements of a stressed skin fuselage are (i) the skin, (ii) the longitudinal stringers and (iii) the frames or bulkheads. The frames serve the same purpose as the ribs in a wing. They distribute local loads to the skin, they stabilise the stringers, and they assist in maintaining the contour. The skin and stringers may be analysed in the manner suggested in para. 7.2, but the frames, particularly those at which there are local concentrated loads, present some difficulty. Various simplifying assumptions are often made when a frame is stressed, but even then the analysis of any frame other than a circular frame is an awkward problem. The stressing of fuselage frames is beyond the scope of this book, and we shall state merely that any concentrated loads applied to a frame are resisted by shear flow in the skin, the shear flow being not necessarily uniform round the section.

Holes cut in the skin for doors or windows give rise to other problems. In order to prevent undue distortion, reinforcement must be added around the cut-out, capable of transferring the shear with no more distortion than if the skin were continuous. In the region of a cut-out, those longitudinal stringers which are cut are not fully effective up to the cut-out, and it is often assumed that they are completely ineffective between such a cut-out and the next frame.

14.11. Tail Unit Stressing

For the tailplane and elevator, and for the fin and rudder, the spanwise distribution of load is generally assumed to be proportional to the chord. This distribution is modified if the fins are located at the tips of the tailplane. The chordwise load distribution is prescribed in A.P. 970, and varies with the centre of pressure position. The general procedure for stress analysis is similar to that already given for the wing. Concentrated loads will be applied at the elevator

hinges to the tailplane, and at the rudder hinges to the fin, due to the loading on the control surfaces. Account should be taken of the fact that the tailplane and fin are not rigid structures when determining the elevator and rudder hinge loads.

14.12. Undercarriage Stressing

The methods of analysis to be adopted here depend on the type of undercarriage. The truss type of undercarriage will be analysed as a space frame, whilst the cantilever or semi-cantilever type will require analysis as a beam subjected to combined shear, bending and torsion. It may be necessary to use the theory for curved beams (Chapter IV) for some parts of the undercarriage structure, notably the wheel forks.

14.13. Structural Testing

Only rarely is it possible to rely entirely on calculations in the design of an aeroplane structure, and it is usual to check the strength and stiffness of the major structural components by means of tests. Often, too, *ad hoc* tests are carried out on sub-components, such as skin-stringer panels, in order to furnish design data.

The major problem in structural testing is the reproduction, as closely as possible, of the loading system assumed in the theoretical design. Many loading methods have been evolved, ranging from the use of bags, filled with sand or lead shot, which were piled on the component under test in such a manner as to approximate to the assumed distribution of load, to complex linkage systems, loaded either mechanically or hydraulically. Test frames are in existence nowadays which enable strength tests to be carried out on full-span wings, which are attached to part of the fuselage structure, in order to reproduce the actual conditions at the supports. The desired distribution of load over the wing is obtained by means of a large number of loading points, which are attached to a linkage system to reduce the number of external loads to be applied. The Cathedral test frame, at the Royal Aircraft Establishment, is of this type.

There are two basic ways of applying load to the component under test, illustrated diagrammatically in Fig. 14.12. There are, of course, many variants of these, but the fundamental difference between the two types is that in case (a) the applied load W constitutes an additional load to be carried by the floor, whilst in case (b) the applied load and the reaction provided by the test component balance out internally in the closed circuit formed by the frame, the loading linkage (including straining device), and the test component, so that no extra load comes on to the floor. Most modern test frames utilise the principle of the closed circuit.

In the case of a test on a very large component, it is not possible to link all the loading points to one final applied load, and a number of separate external loads have to be applied. The problem of ensuring that all these external loads are applied at the correct rate, maintaining the same proportionality between them, is one which has been given considerable attention. It is not possible to discuss the various methods of loading which have been adopted in an attempt to overcome the many difficulties, and the student is referred to papers which have appeared in technical journals for more detailed information. (A number of such papers is listed at the end of this chapter.)

A very necessary part of the procedure in carrying out a structural test is the measurement of deflections. The development of the electrical resistance strain

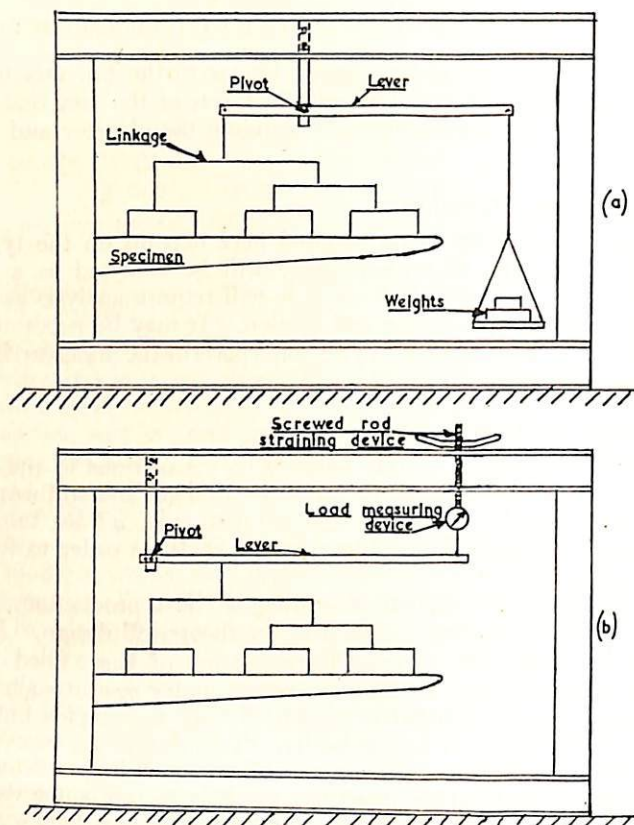


FIG. 14.12.—ALTERNATIVE METHODS OF APPLYING LOAD TO A TEST SPECIMEN.

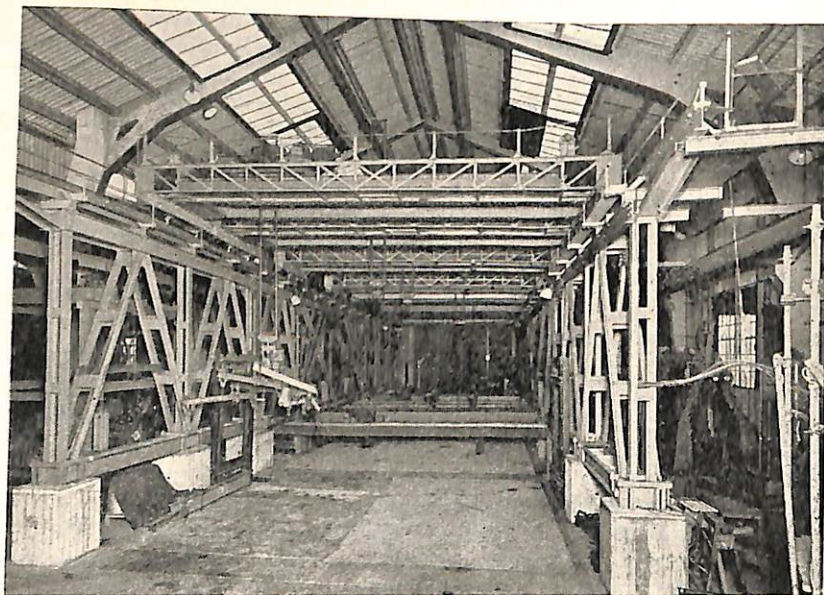


FIG. 14.13.—THE CATHEDRAL TEST FRAME AT THE ROYAL AIRCRAFT ESTABLISHMENT, SHOWING BEGINNINGS OF TEST RIG BEING ASSEMBLED.

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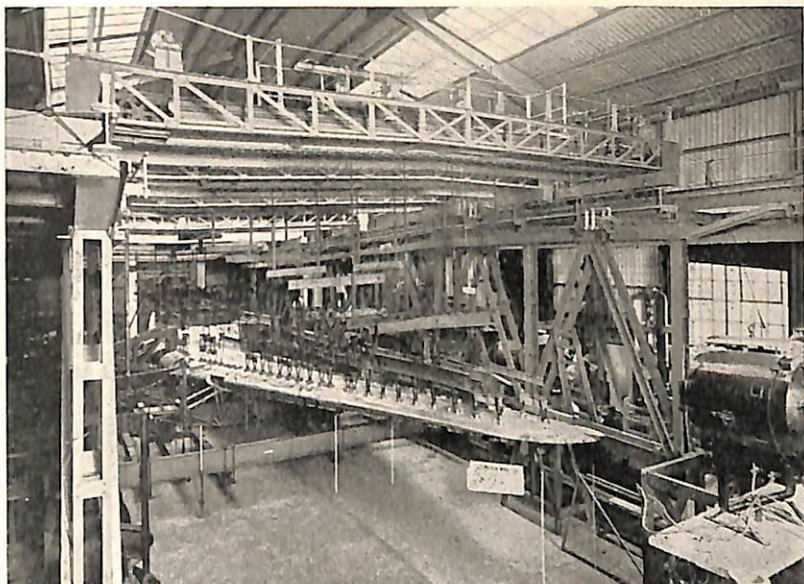


FIG. 14.14.—FULL SPAN WING RIGGED FOR TEST IN CATHEDRAL TEST FRAME.

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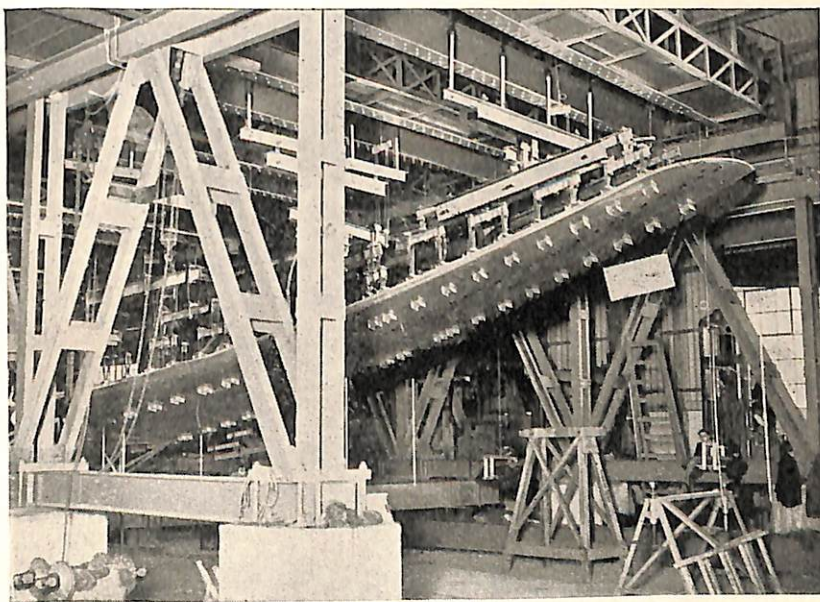


FIG. 14.15.—WING OF FIG. 14.14 JUST BEFORE FAILURE. NOTE THE LARGE TIP DEFLECTION.

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gauge has enabled detailed measurements of strain to be made, and it is now possible to obtain a reasonably complete idea of the distribution of strain, and hence, for convenience, of stress in a component under load. (See Ref. 14.8.)

References for Further Reading

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APPENDIX I

GRAPHICAL METHODS

A.I.1. In this section are given some graphical methods of evaluating certain quantities, which may be of value on occasions when analytical methods become cumbersome or impracticable. Only methods which are considered to be of particular value in aircraft structural analysis are given, and it is not suggested that others, not included, are not of use. Such items as the various rules for determination of areas are not mentioned, it being assumed that the student will be familiar with these.

A.I.2. Determination of First Moment of Area of a Thin-walled Section

To determine the moment of area of the thin strip ABCD about the axis XX, a convenient method is as follows. Divide the projected length of the strip on XX into a number of elements and erect ordinates as shown in Fig. A.I.1. Construct a curve of "squared ordinates"—i.e., locate such points as E' and F' by

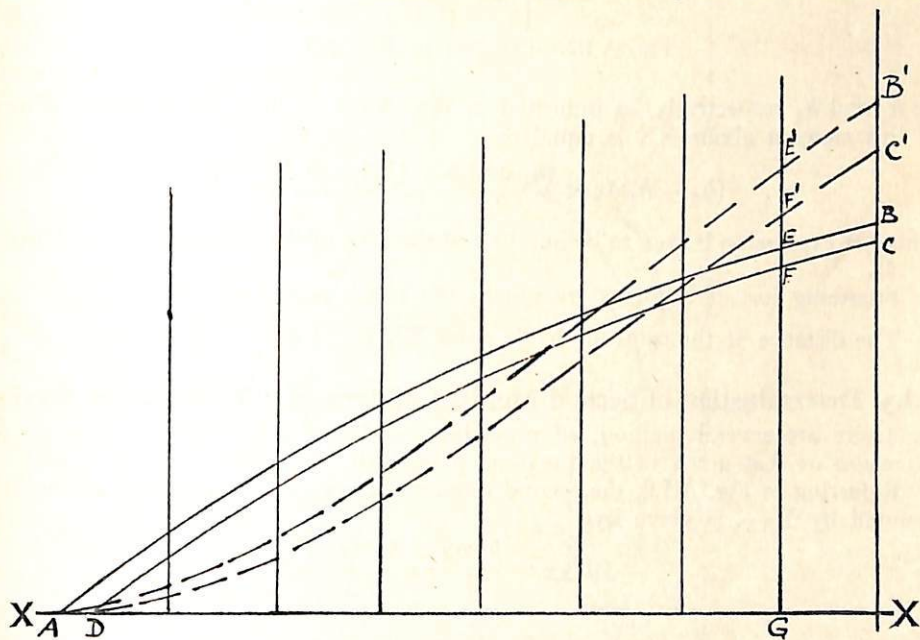


FIG. A.I.1.—GRAPHICAL DETERMINATION OF FIRST MOMENT OF AREA OF THIN-WALLED SECTION BY MEANS OF A CURVE OF SQUARED ORDINATES.

making $E'G = (EG)^2$ and $F'G = (FG)^2$. Let the area of this new curve be A_2 . Then the moment of the area, A_1 , of the strip ABCD about XX is equal to $\frac{1}{2}A_2$.

This may be proved as follows. Consider an element of ABCD, of width δx , this width being small enough for the element to be regarded as a rectangle. Let the ordinates, measured from XX, of the lower and upper edges of the element

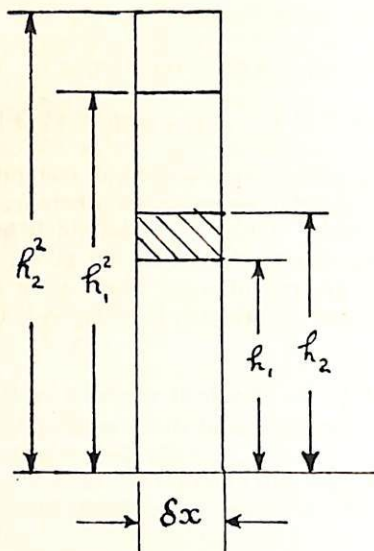


FIG. A.I.2.—ELEMENT OF FIG. A.I.1.

be h_1 and h_2 respectively, as indicated in Fig. A.I.2. Then the moment of area of this element about XX is equal to

$$(h_2 - h_1)\delta x \times \frac{(h_1 + h_2)}{2} = \frac{(h_2^2 - h_1^2)}{2}\delta x$$

This last expression is seen to be one-half of the area of the corresponding element of A_2 .

Summing for all elements we obtain the result stated above.

The distance of the centroid of A_1 from XX is, of course, given by $\frac{1}{2} \times \frac{A_2}{A_1}$.

A.I.3. Determination of Second Moment of Area of a Thin-walled Section

There are several methods of procedure, and the one described here is an extension of that given in the previous paragraph.

Referring to Fig. A.I.2, the second moment of area of the element about XX denoted by δI_{XX} , is given by

$$\begin{aligned} \delta I_{XX} &= \frac{h_2^3 \delta x}{3} - \frac{h_1^3 \delta x}{3} \\ &= \frac{(h_2^3 - h_1^3) \delta x}{3} \\ &= \frac{1}{3} (\text{area of corresponding element of curve of} \\ &\quad \text{"cubed ordinates"}) \end{aligned}$$

Hence, if we construct a curve whose ordinates, measured from XX, are the cubes of the ordinates at corresponding points of the original strip ABCD, and if the area of this curve is A_3 , then

$$I_{XX} \text{ of ABCD} = \frac{A_3}{3}$$

A.I.4. Determination of Z for Curved Beams. (See para. 4.4)

The quantity Z for a curved beam is defined by equation 4.33 as

$$Z = R \int \frac{1}{R+y} dA - A$$

where R is the radius of curvature of the centroidal axis, y is the distance of a point in the cross-section measured from the centroidal axis, and A is the area of the cross-section. A convenient graphical method of evaluating the integral in the expression for Z is the following:—

Consider the section shown in Fig. A.I.3. The centroid of the section is at G , and the centre of curvature of the centroidal surface is at O , so that $OG = R$.

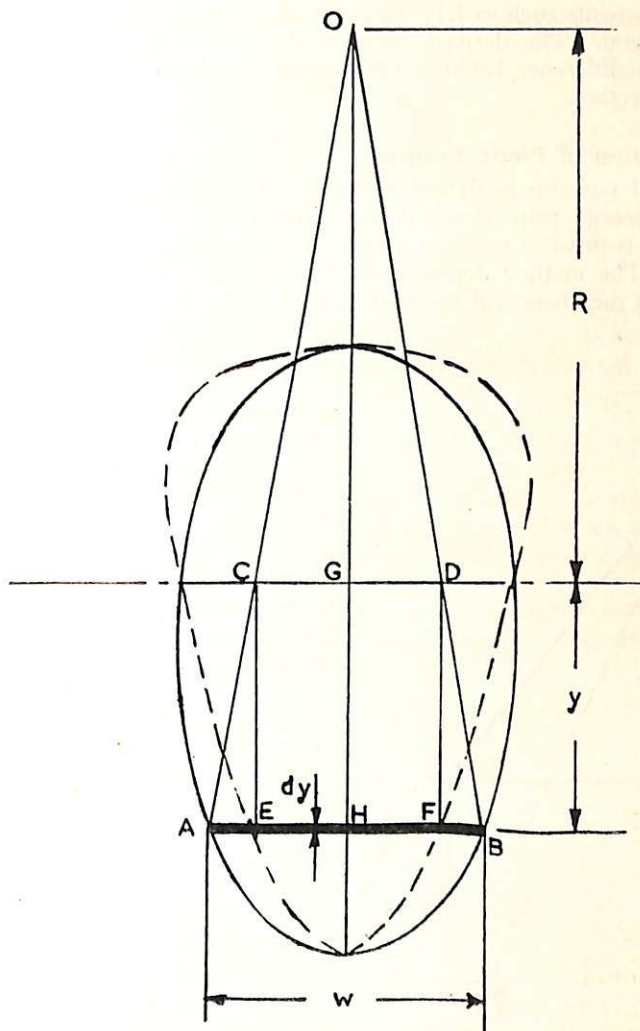


FIG. A.I.3.—GRAPHICAL DETERMINATION OF Z FOR A CURVED BEAM.

Let an element AB, at distance y from the centroidal axis, be of width w and thickness dy . Then the area of this element, dA , is equal to $w dy$.

Let the lines joining O to the ends A and B of this element cut the centroidal axis in C and D respectively, and draw CE and DF perpendicular to AB, as shown in the figure. Then we see that

$$\frac{EF}{AB} = \frac{CD}{AB} = \frac{GO}{OH} = \frac{R}{R+y}$$

Hence, the area of the element EF is

$$EF \times dy = \frac{R w dy}{R+y} = \frac{R dA}{R+y}$$

Consequently, the area of a curve derived by transforming all elements such as AB into elements such as EF, using the above procedure, will be equal to the required integral. The derived curve is shown dotted in the figure. Z then follows as the difference between the area of the derived curve and the area of the original section.

A.I.5. Deflection of Plane Frames

Chapter II contains analytical methods for evaluating deflections of frames, using strain energy principles. If the loads in the members of a plane frame are known, it is possible to use a graphical construction, if desired, to determine deflections. The method depends on the fact that extensions and contractions of the various members will be small, and it is assumed that the elastic limit is not exceeded (although this last assumption is not essential, so long as the changes in length of the members are known and are small). The basic principles of the method may be explained by reference to Fig. A.I.4, showing a simple frame ABC, pinned to a rigid wall at A and B, and carrying a load W at C. Obviously the member AC will be in tension, and will extend, whilst the member BC will be in compression and will contract. If the extended length of AC is AC_1 , then the deflected position of C must lie somewhere on the arc C_1C_3 of the circle whose centre is A and whose radius is AC_1 . Similarly, if the contracted length of BC is BC_2 , the deflected position of C must lie also on the arc C_2C_3 of the circle with centre B and radius BC_2 . The new position of C is thus defined by the point of intersection, C_3 , of these two arcs. The distances CC_1 and CC_2 will be very small compared with AC and BC, and it would be very difficult to draw a diagram as suggested on a scale sufficiently large to enable the deflection to be measured. We can, however, utilise the

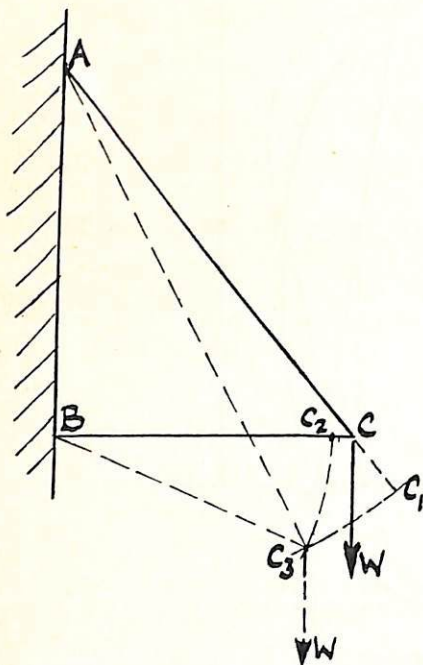


FIG. A.I.4.—DEFLECTION OF SIMPLE PLANE FRAME.

fact that the deformations are small by remarking that the arcs C_1C_3 and C_2C_3 will approximate very closely to straight lines perpendicular to AC and BC respectively, and we then adopt the following construction.

In Fig. A.I.5, the point a,b represents the fixed points A and B of the frame. The line aC_1 represents, to a suitable scale, the extension of AC , and the line bC_2 represents, to the same scale, the contraction of BC . These two lines are drawn in the directions in which the point C moves along AC and BC relative to the fixed points A and B . Perpendiculars to aC_1 and bC_2 from C_1 and C_2 respectively intersect in C_3 . Then this diagram is a reproduction of the portion $CC_1C_3C_2$ of Fig. A.I.4, and the horizontal and vertical displacements of C_3 from

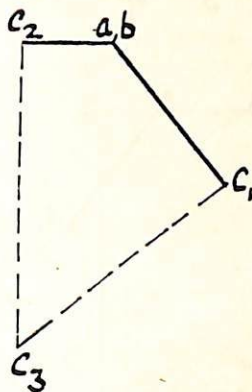


FIG. A.I.5.—GRAPHICAL DETERMINATION OF DEFLECTION OF JOINT OF A PLANE FRAME.

a,b represent the horizontal and vertical components of the deflection of the point C in the frame.

By repeated application of this procedure, it is possible to obtain a diagram (Fig. A.I.6) which gives the deflection of all the joints of a frame such as that shown in Fig. 2.4. The construction of this diagram, known as a Williot diagram, is as follows.

Using the loads determined in Example 2.1, we calculate first the extensions and contractions of the several members. These changes in length will be given

by $\frac{PL}{AE}$ for each member. Thus we have

Member.	P (lb.)	$\frac{PL}{AE}$ (in.).
AB	750	5.4×10^{-3}
BC	1250	15.0×10^{-3}
CD	— 750	$- 5.4 \times 10^{-3}$
BD	— 1000	$- 9.6 \times 10^{-3}$
DE	— 1500	$- 10.8 \times 10^{-3}$
DA	1250	15.0×10^{-3}

The points A and E are fixed, so we start from a point a,e on our diagram,

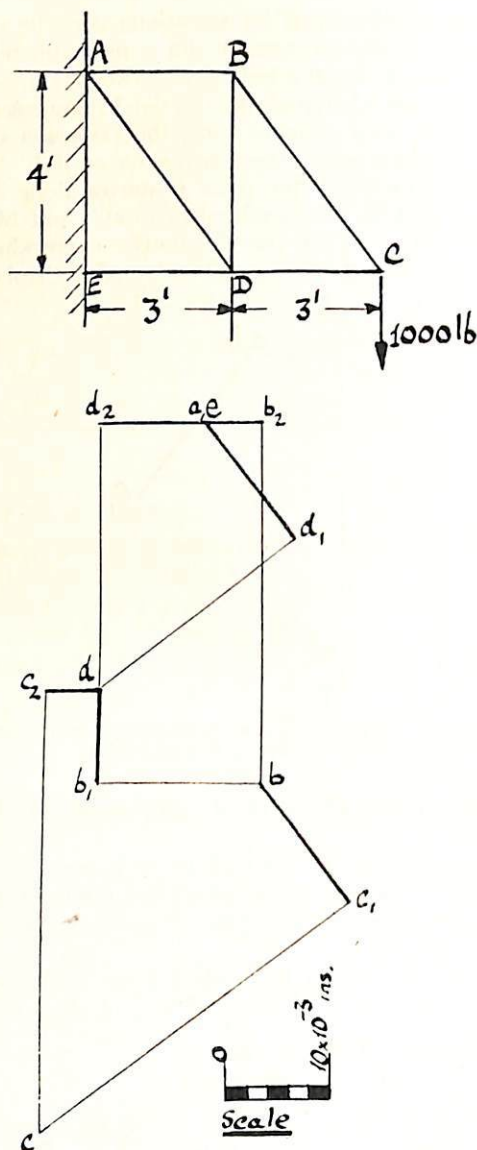


FIG. A.I.6.—DEFLECTION OF A PLANE FRAME. CONSTRUCTION OF WILLIOT DIAGRAM.

and locate first the deflected position of D relative to A and E. To do this, ad_1 and ed_2 are drawn to represent the movements of D relative to A and E along AD and ED. The point d, the intersection of the normals to ad_1 and ed_2 from d_1 and d_2 , represents the deflected position of D relative to A and E. We can now start from a and d to locate the point b, corresponding to the deflected position of B. Since the member DB is in compression, B will move downwards

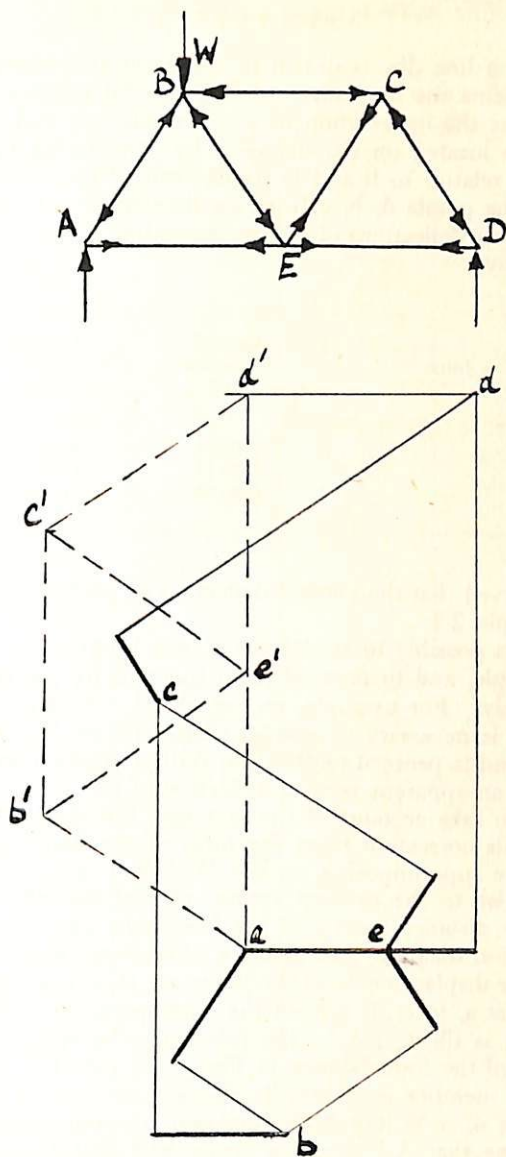


FIG. A.I.7.—MOHR CORRECTION TO WILLIOT DIAGRAM.

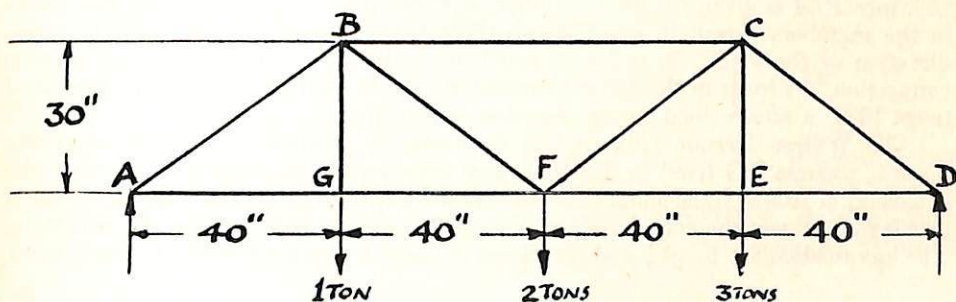


FIG. A.I.8.—NUMERICAL EXAMPLE FOR APPLICATION OF WILLIOT-MOHR DIAGRAM.

relative to D, and a line db_1 is drawn to represent this movement. Similarly, the line ab_2 represents the movement of B along AB relative to A. The point b is then located as the intersection of the normals b_1b and b_2b . In a similar way the point c is located on the diagram, by considering the movement of C along BC and CD relative to B and D respectively. The vertical and horizontal displacements of the points d, b, c from a,e then represent, to the chosen scale, the components of the deflections of the corresponding points in the frame. From the diagram we have

Joint.	Vertical Deflection (in.).	Horizontal Deflection (in.).
B	0.0364	0.0054 →
C	0.0714	0.0162 ←
D	0.0268	0.0108 ←

It will be observed that the vertical deflection of joint C agrees with the value obtained in Example 2.1.

It is not always possible to start from a base point such as the point a,e of the previous example, and to proceed from there to fix the displacement of the next joint absolutely. For example, in Fig. A.I.7, we can start from A as our fixed point, but it is necessary to assume either AE or AB fixed in direction in the first instance, and to proceed to draw the Williot diagram with this assumption. This will result in an apparent vertical deflection of the point D, and a correction must be applied to take account of the fact that the vertical deflection of D is actually zero. This correction takes the form of the Mohr correction diagram, and is obtained by superimposing on the Williot diagram, a diagram which is geometrically similar to the original frame, rotated through, in this case, 90° . The perpendicular distance from a to the horizontal through d is the base line ad' of this correction diagram, and $ab'c'd'e'$ then reproduces the frame ABCDE to scale. The true displacements of the joints are then measured as the displacements of the points a, b, c, d, e from the corresponding points a, b', c', d', e'.

The procedure is illustrated by the following example:—

The members of the frame shown in Fig. A.I.8 are of such cross-section that the stress in each member is 20,000 lb. per sq. in., and $E = 30 \times 10^6$ lb. per sq. in. By means of a Williot-Mohr diagram, determine the deflections of all the joints, assuming that A is fixed in space, and that D is free to move only horizontally.

Since P/A is given for each member, it is not necessary to evaluate the loads in the members except in cases where there may be some doubt regarding the direction of the load. In order to avoid any difficulty which may arise in this connection, the loads in the various members have been determined and tabulated (page 198), a tensile load being regarded as positive.

The Williot diagram (Fig. A.I.9) is drawn as follows:—Starting from the point a, assume AG fixed in direction and draw ag to the right to represent the extension of AG to some suitable scale. Draw ab_1 downwards to the left, parallel to AB to represent the contraction of AB on the same scale and draw gb_2 similarly. The perpendiculars to gb_2 and ab_1 from b_2 and b_1 respectively intersect in b.

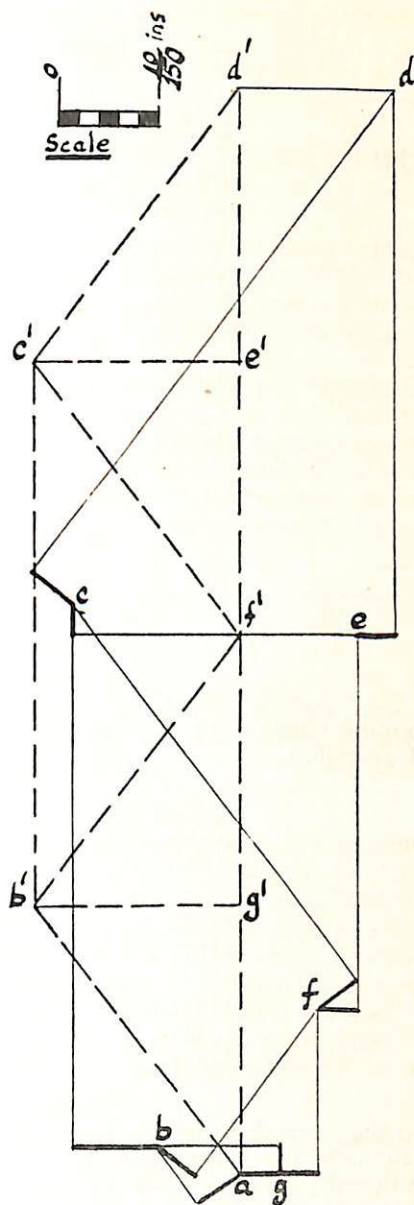


FIG. A.I.9.—WILLIOT-MOHR DIAGRAM FOR FRAME OF FIG. A.I.8.

Next locate the point f , from g and b in a similar manner, and proceed in the same way until the Williot diagram is completed.

ad' is now drawn perpendicular to the horizontal through d , and the Mohr correction diagram is drawn on this line as base. By measuring the required

Member.	Load (tons).	Length (in.).	PL/AE (in.).
AB	$-\frac{12.5}{3}$	50	$-\frac{5}{150}$
BC	$-\frac{16.0}{3}$	80	$-\frac{8}{150}$
CD	$-\frac{17.5}{3}$	50	$-\frac{5}{150}$
DE	$+\frac{14.0}{3}$	40	$+\frac{4}{150}$
EF	$+\frac{14.0}{3}$	40	$+\frac{4}{150}$
FG	$+\frac{10.0}{3}$	40	$+\frac{4}{150}$
GA	$+\frac{10.0}{3}$	40	$+\frac{4}{150}$
BG	$+1.0$	30	$+\frac{3}{150}$
BF	$+\frac{7.5}{3}$	50	$+\frac{5}{150}$
FC	$+\frac{2.5}{3}$	50	$+\frac{5}{150}$
CE	$+3.0$	30	$+\frac{3}{150}$

distances from the appropriate points on the correction diagram, the desired deflections are obtained as under.

Joint.	Vertical Deflection (in.).	Horizontal Deflection (in.).
B	0.1622 ↓	0.08 →
C	0.1622 ↓	0.0267 →
D	0.0	0.1067 →
E	0.1822 ↓	0.08 →
F	0.2533 ↓	0.0533 →
G	0.1822 ↓	0.0267 →

It may be noted that the correction diagram is actually a diagram giving the displacements, relative to A, of the joints when the frame is rotated about A, without deformation, so that the joint D moves vertically a distance ad' , and the Mohr correction is merely the superimposing of these deflections on to those determined from the Williot diagram, in which AG is assumed fixed in direction.

APPENDIX II

NOTE ON THE HARDY CROSS METHOD OF MOMENT DISTRIBUTION

A.II.1. Introduction. Sign Convention

WHEN the number of bays of a continuous beam is large, the number of simultaneous equations resulting from application of the three moment equation is also large, and the solution is apt to become somewhat laborious. The work can be lessened by use of the moment distribution method, developed by Professor Hardy Cross. This brief introduction to the use of the method is given in an appendix rather than in the body of the text because it is convenient to adopt a sign convention which differs from that used elsewhere in this book. This convention relates to the sign of the *end moments* applied to a beam, and refers the sign to the direction in which the moment tends to rotate the beam. In this account of the moment distribution method, we shall define a *positive* end moment as one which tends to rotate the beam in a counter-clockwise direction.

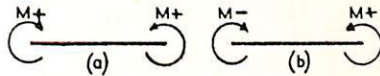


FIG. A.II.1.—SIGN CONVENTIONS: (a) BEAM THEORY; (b) MOMENT DISTRIBUTION.

By adopting this convention, we have a positive moment producing a positive slope. The difference between this sign convention and that used earlier in our treatment of beam theory is illustrated in Fig. A.II.1.

A.II.2. General Principles of Method

In applying this method to the solution of a continuous beam, the first step is to assume each bay of the beam to be built in at its supports, and to calculate the resulting end moments. This will generally result in the moments on the two sides of any support being of different values, so that there is an unbalanced moment at each support, whose value is easily determined. The supports are then released, by the application of a balancing moment at each support, and the effects of adding these extra moments is assessed. The procedure is continued until the moments at all the supports are balanced. Numerical examples will make the procedure clear, after the effects of the additional moments have been discussed.

A.II.3. Distribution Factors and Carry-over Moments

If a beam ABC, built in at A and C, is continuous over the support B, a moment M applied at B will be shared between the two bays AB and BC in such a way that the slope at B is the same for both bays. If the moment taken by AB is M_1 and that taken by BC is M_2 , it may be verified quite easily that the relationship between M_1 and M_2 is

$$\frac{M_1}{M_2} = \frac{4E_1I_1}{L_1} \bigg/ \frac{4E_2I_2}{L_2} \quad \dots \quad \text{Eq. A.II.1}$$

where the suffix 1 refers to the bay AB, and the suffix 2 to BC. The quantity

$4EI/L$ is a measure of the resistance of the beam to rotation, and is sometimes called the *stiffness factor*. The *distribution factors* are then $M_1/(M_1 + M_2)$ for AB and $M_2/(M_1 + M_2)$ for BC.

It may be verified also that if a moment M_1 is applied at B to a beam AB, which is built in at A and simply supported at B, the moment produced at A will be equal to $M_1/2$. With our new sign convention for end moments, the *carry-over moment* thus produced at A will be of the same sign as the moment applied at B.

A.II.4. Numerical Illustrations

In order to illustrate the application of the method to continuous beams, the examples of Chapter IX will be worked out.

Example A.II.1. Determine the fixing moments at the supports for the beam of Example 9.2, by the moment distribution method.

Since the beam is uniform, the stiffness factors of the three bays will be proportional to $1/L$ in each case.

The distribution factors are evaluated as follows:—

$$\text{At B, the distribution factor for AB} = \frac{1}{4} / \left(\frac{1}{4} + \frac{1}{5} \right) = \frac{5}{9}$$

$$\text{“ “ “ “ BC} = \frac{4}{9}$$

$$\text{At C, “ “ “ “ CB} = \frac{6}{11}$$

$$\text{“ “ “ “ CD} = \frac{5}{11}$$

These distribution factors give the proportions in which a moment applied at B or C is to be shared between the two bays meeting at the support.

The end moments are now calculated, assuming that each bay is completely fixed at both ends. Thus, for AB, reference to equation 9.6 shows that the fixed end moments are equal to $\frac{360 \times 4^2}{12} = 480$ lb. ft. Following our sign convention,

the moment will be positive at A, and negative at B. The fixed end moments are entered in the first line of the tabular calculation below. In the second line of this table, the moments required to balance at each support are applied in the predetermined proportions to the bays. For instance, at B the unbalanced moment is + 145 lb. ft., so that - 145 must be applied for balance. Of this moment, $\frac{5}{9} \times 145 = 81$ is applied to AB, and $\frac{4}{9} \times 145 = 64$ is applied to BC. The

moments at A and D must be balanced out since these supports are simple supports, which can supply no fixation. The third line of the table shows the carry-over moments resulting from the application of the balancing moments. A second balance is now required, with further carry-over moments, and the process is continued until the carry-over moments become small enough to be neglected. Addition of the moments applied at each support then gives the total fixing moments, which will be seen to agree with the results obtained in Example 9.2. In this case, the calculations could have been terminated after the fifth balance, as the carry-over moments have become quite small, but the table has been continued beyond this point in order to show the method in detail. The actual

working in the table could be reduced by taking account of the simple supports at A and D in the first place. This can be done by taking $\frac{3}{4}$ of the stiffness factor for the bays AB and CD, and omitting any carry-over of balancing moments to A and D. The initial fixed end moments for these two bays will in this case be calculated for the one end fixed condition, and it can be verified that the initial moments for this example will then be, reading from left to right in the table from A to D, zero at A, -720 and $+625$ at B, -625 and $+1080$ at C, zero at D. The procedure is then exactly the same, except that the revised distribution factors are used, and there is no carry-over to A or D.

Distribution Factors.	A	B		C		D
		5/9	4/9	6/11	5/11	
Fixed end moments	$+480$	-480	$+625$	-625	$+720$	-720
First balance	-480	-81	-64	-52	-43	$+720$
First carry-over	-40	-240	-26	-32	$+360$	-21
Second balance	$+40$	$+148$	$+118$	-179	-149	$+21$
Second carry-over	$+74$	$+20$	-89	$+59$	$+10$	-74
Third balance	-74	$+38$	$+31$	-38	-31	$+74$
Third carry-over	$+19$	-37	-19	$+15$	$+37$	-15
Fourth balance	-19	$+31$	$+25$	-28	-24	$+15$
Fourth carry-over	$+15$	-9	-14	$+12$	$+7$	-12
Fifth balance	-15	$+13$	$+10$	-10	-9	$+12$
Fifth carry-over	$+6$	-7	-5	$+5$	$+6$	-4
Sixth balance	-6	$+7$	$+5$	-6	-5	$+4$
Sixth carry-over	$+3$	-3	-3	$+2$	$+2$	-2
Seventh balance	-3	$+3$	$+3$	-2	-2	$+2$
Total	0	-597	$+597$	-879	$+879$	0

Example A.II.2. Applying this method to the beam of Example 9.3 gives a very rapid solution. A cantilever, such as the bay BC, has zero stiffness factor, so that the distribution factors at B are 1.0 for AB and 0 for BC. Further, since the beam is built in at A, the moment here does not require to be balanced out. Thus, one balance and one carry-over complete the solution, as shown in the table.

Distribution Factors.	A	B		C
		1.0	0	
Fixed end moments	$+13333$	-13333	$+2000$	0
First balance		$+11333$	0	
First carry-over	$+5667$	0	0	0
Total	$+19000$	-2000	$+2000$	0

A.II.5. Other Applications

The method of moment distribution has been considered above only in relation to continuous beams on rigid supports. The method can be extended to deal with continuous beams on supports which deflect, and it is of great use in connection with the analysis of rigid jointed frames. No further applications will be considered here, the purpose of this brief discussion being merely to serve as an introduction to the method.

APPENDIX III

DEFINITIONS OF FLEXURAL CENTRE, SHEAR CENTRE AND FLEXURAL AXIS

A.III.1. General

IN Chapters V, VII and XIV reference is made to the *shear centre* of a shell which is subjected to combined bending and torsion (e.g., a wing). This point has been defined as the point in the cross-section at which a transverse load must be applied so as to produce pure bending without twist. *Except for a constant section shell*, this definition is meaningless unless we define twist more precisely. Furthermore, considerable confusion exists regarding the definitions of shear centre and flexural centre and flexural axis, and a statement of the definitions as they relate to modern usage is considered desirable. Definitions of flexural centre and flexural axis other than those given below may be found in older literature, and it is perhaps unfortunate that the same terms have been retained as the changes in aircraft structure have brought the need for revision of the definitions.

A.III.2. Definitions

The flexural centre of a wing with reference to a particular section is the point in that section at which a transverse load must be applied so as to produce *zero twist of that section relative to the root*.

The flexural axis of a wing is the line through the flexural centre perpendicular to the plane of symmetry (or root plane).

The shear centre of a wing section is the point on the section at which a load must be applied so as to produce *zero rate of change of twist (or zero twist per unit length) at that section*.

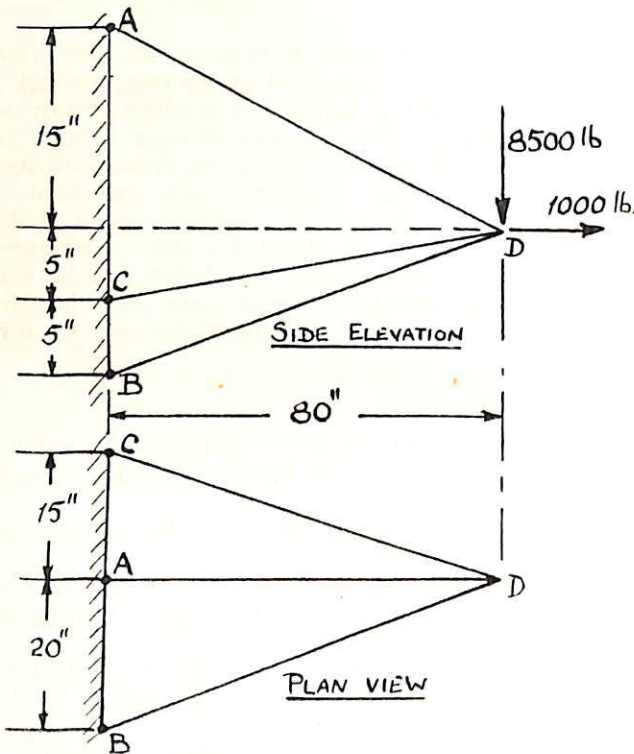
It will be observed that the flexural centre and flexural axis are *characteristics of the complete wing with reference to a particular section*, whilst the shear centre is a *characteristic of a particular section alone*.

In structural analysis the shear centre is the important point. The location of the flexural centre, as defined above, may be required for the determination of the wing stiffness, and, as such, is primarily of experimental importance.

A more general survey of these definitions is contained in the author's paper "Shear Centre, Flexural Centre and Flexural Axis," published in *Aircraft Engineering*, July 1951:

EXAMPLES

1. The tripod structure in the figure is loaded as shown. Find the reactions at A, B and C and the loads in the members. (R.Ae.S.)



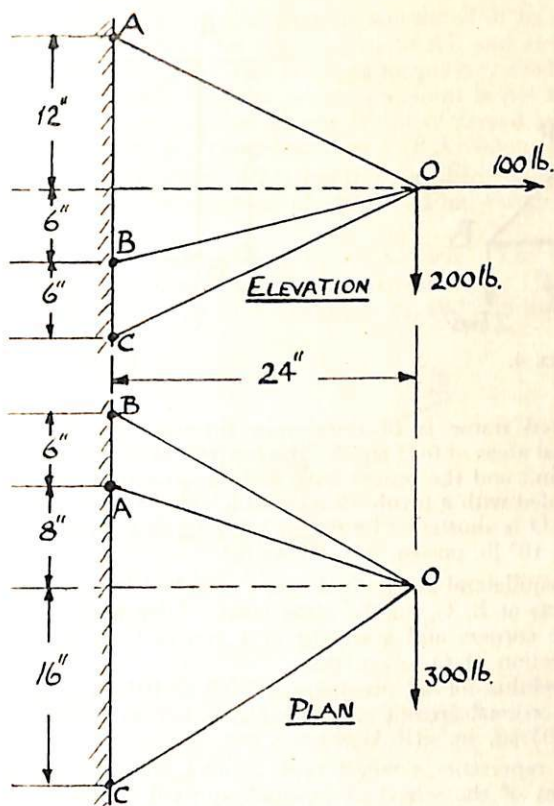
EXAMPLE 1.

2. Use the method of tension coefficients to find the loads in the three dimensional framework shown under the given applied loads. (R.Ae.S.)

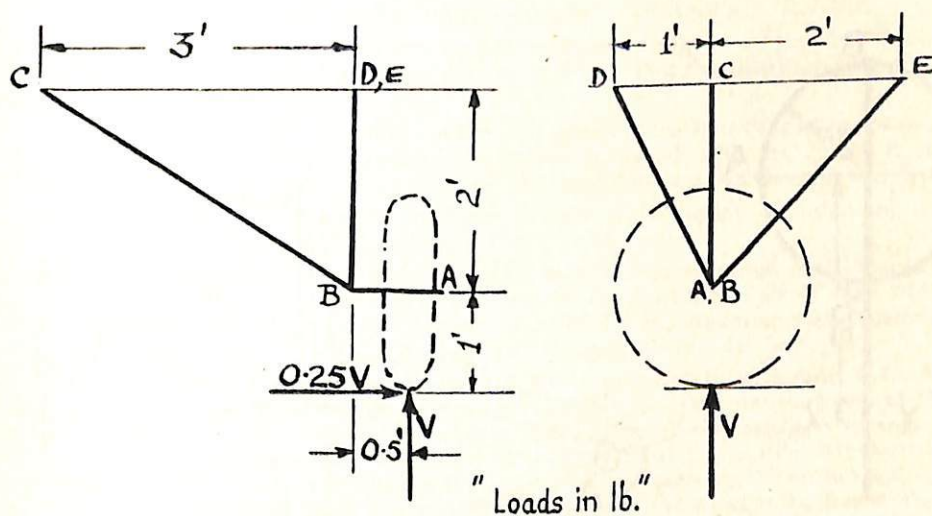
3. The diagram represents a tripod type undercarriage, with vertical and horizontal loads applied at the point of tyre contact. The member ABC is continuous and is pin-jointed to the aeroplane structure at C. Members DB and EB are pinned to ABC and to the aeroplane at D and E. Calculate the end load in each member, and the bending moment in ABC at point B, in terms of load V . (R.Ae.S.)

4. State the theorem you would use to find the loads in a redundant structure (not self strained).

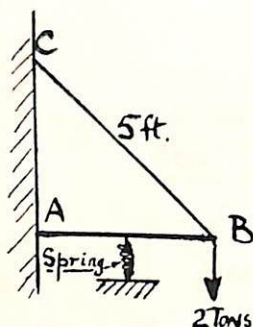
In the figure AB is a light rigid bar pin-jointed to a wall at A and it rests horizontally on a spring at its mid-point. A wire BC, 5 ft. long, joins B to the wall at a point C, such that $AB = AC$. Find the load in the wire if a two ton weight is hung from B, assuming that the spring deflects 0.01 in. per ton load, and that the cross-sectional area of the wire is 0.1 sq. in. Take $E = 10^7$ lb. per sq. in. (R.Ae.S.)



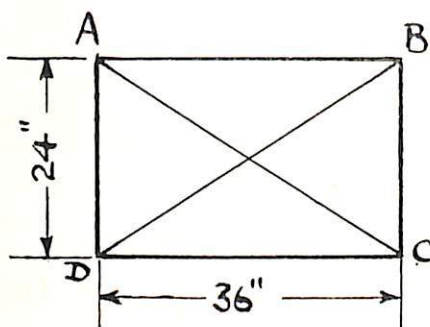
EXAMPLE 2.



EXAMPLE 3.



EXAMPLE 4.



EXAMPLE 5.

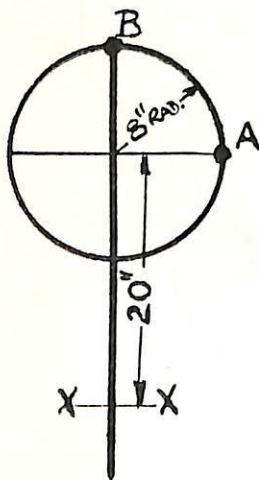
5. A pin-jointed frame is of duralumin throughout. The vertical members have cross-sectional areas of 0.25 sq. in. , the horizontal members have cross-sectional areas of 0.15 sq. in. , and the braces have cross-sectional areas of 0.05 sq. in. The brace BD is provided with a turnbuckle by which it can be tensioned. After assembly the member BD is shortened by 0.1 in. Determine the stresses in the members. Take $E = 10.5 \times 10^6 \text{ lb. per sq. in.}$ (R.Ae.S.)

6. ABC is an equilateral pin-jointed frame of side 5 ft. resting in a vertical plane on smooth supports at B, C, on the same level. Members OA, OB, OC join the centroid O to the corners and a weight of 1 ton is hung at O.

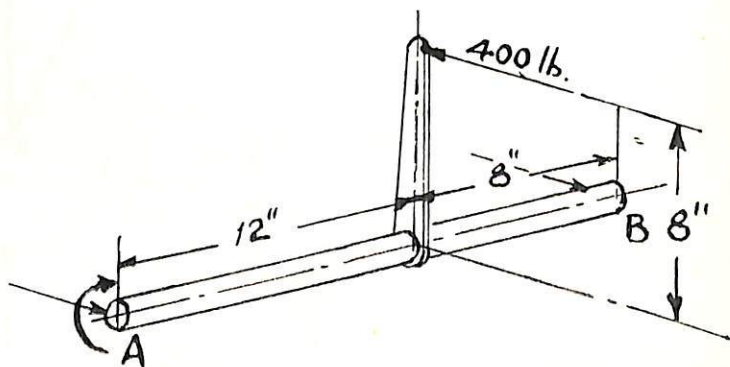
Find the deflection of O, given that

- (i) Young's modulus for all members is $10.2 \times 10^6 \text{ lb. per sq. in.}$
- (ii) The cross-sectional area of AB, BC, CA is 0.1 sq. in. , and that of OA, OB, OC is 0.05 sq. in. (R.Ae.S.)

7. The figure represents a wheel type control column. A load of 200 lb. is applied to the rim of the wheel at point A normal to its plane. Calculate the maximum principal stresses (direct stress and shear stress) at a section XX. At this section the control column is a circular tube 1.5 in. outside diameter and 0.028 in. thick. (R.Ae.S.)



EXAMPLE 7.

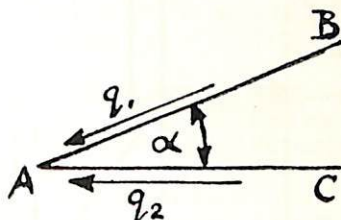


EXAMPLE 8.

8. A circular tube AB, 1 in. in diameter, is supported at its ends A and B in loose bearings. A load of 400 lb., at right-angles to AB and at 8 in. radius from it, is applied to AB through a lever. The resulting torque is reacted at A. Assuming that the maximum direct stress (tension or compression) is not to exceed 20 tons per sq. in. and that the maximum shear stress is not to exceed 12 tons per sq. in., calculate the minimum necessary tube thickness. (R.Ae.S.)

9. A tube 2 in. diameter whose walls are 0.02 in. thick is under the action of a torque of 100 lb. in. and a tensile load of 150 lb. Find the principal stresses in the material. (R.Ae.S.)

10. The figure shows the apex of a wedge of angle α . The sides AB and AC are under the action of shearing stresses q_1 , q_2 respectively. By considering the equilibrium of a small element at A show that $q_1 = q_2$, and find the principal stresses at A. (R.Ae.S.)



EXAMPLE 10.

11. A test specimen consisting of aluminium alloy bar of diameter $\frac{3}{8}$ in. is found to extend by 0.004 in. on a 2-in. gauge length under a tensile load of 1 ton. Assuming Poisson's ratio to be $\frac{1}{4}$, find the angle through which a 3 in. length of the bar would twist under a torque of 200 lb. in. (R.Ae.S.)

12. A strip of metal sheet 1.5 in. wide, 0.048 in. thick, is found when tested under a load of 1,500 lb. tension to extend by 0.0013 in. on a 2-in. gauge length and to contract by 0.00025 in. in width. Calculate Young's modulus, the shear modulus, Poisson's ratio, and the bulk modulus, proving any formulae you use, and specifying the dimensions of the values calculated. (R.Ae.S.)

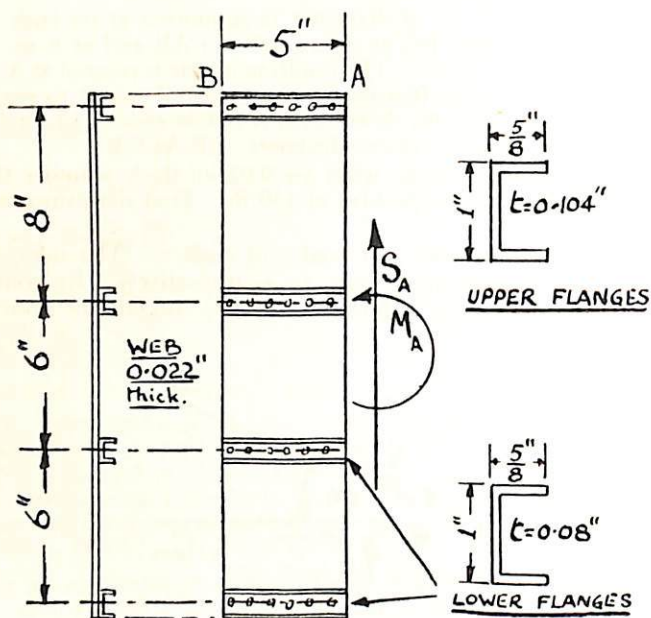
13. A solid bar of circular cross-section used as a beam is subjected to a shear force of 5,000 lb. The diameter of the bar is 2 in. Plot the distribution of shear stress over the section.

14. Determine the shear stresses in the web panels of the four flange beam shown. The shear force and bending moment at station A are 5000 lb. and 25,000 lb. in. respectively, and the shear force may be assumed constant between A and B.

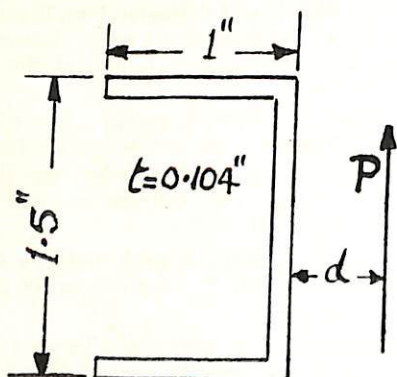
15. For the channel section in the figure determine the distance d so that application of the load P produces bending without twist.

16. The leading edge of a single spar wing may be represented as in the diagram. If the covering and spar web thicknesses are $\frac{1}{16}$ in., find the angle of twist of an 8-ft. length of the wing under a torsion of 2,500 lb. ft., assuming end constraint effects are negligible. Take $G = 0.3 \times 10^6$ lb. per sq. in. (R.Ae.S.)

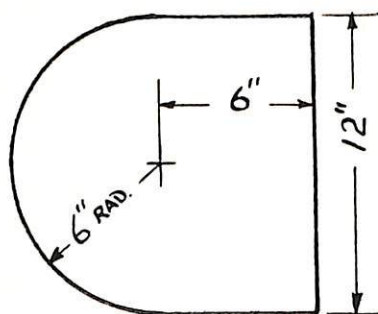
17. A hollow circular cylinder made of thin sheet has to withstand a torque of 5×10^4 lb. ft. If the radius of the cylinder is 6 in., find the thickness of the walls necessary for the sheet to develop a shear stress of 20 tons per sq. in. If the cylinder is 5 ft. long and held fixed at one end, find the couple that must be applied at the other end to produce a twist of one radian between the ends, assuming G to be 4.2×10^6 lb. per sq. in., and the thickness of the sheet to be that already found. (R.Ae.S.)



EXAMPLE 14.



EXAMPLE 15.



EXAMPLE 16.

18. The ordinates of a wing section are as follows :—

Distance from L.E. (in.)	0	1.25	2.5	5.0	7.5	10.0	15.0
Upper surface . . .	0	3.34	4.44	5.89	6.91	7.64	8.52
Lower surface . . .	0	-1.54	-2.25	-3.04	-3.61	-4.09	-4.84

Distance from L.E. (in.)	20	25	30	40	50	60
Upper surface . . .	8.92	9.08	9.05	8.59	7.74	6.61
Lower surface . . .	-5.41	-5.78	-5.96	-5.92	-5.50	-4.81

The spars are located at 25 in. and 60 in. from the leading edge. Using the notation of Fig. 6.7, the skin thicknesses are as follows :— $t_{ac} = 0.048$ in., $t_{ab} = 0.048$ in., $t_{bd} = t_{bf} = 0.028$ in., $t_{be} = 0.036$ in. Determine the shear stresses in the skin panels and spar webs for an applied torque of 150,000 lb. in. Taking G as 4.2×10^6 lb. per sq. in. calculate the angle of twist per foot length.

19. For the section of the previous example the material effective in bending may be assumed concentrated in the spar flanges. The area of each flange of the front spar is 2.0 sq. in., and the area of each flange of the rear spar is 1.5 sq. in. Using the simplified theory of Chapter VII, determine the shear stresses in the skin and in the spar webs due to the application of a load of 2500 lb. at 30 in. behind the front spar.

20. Locate the shear centre of the section of Example 19.

21. A solid circular shaft, 2 in. in diameter, is supported in bearings 4 ft. apart, and has to withstand a lateral load of 5000 lb. at 18 in. from one bearing. Determine the deflection at the point of application of the load, and the maximum deflection. Take $E = 30 \times 10^6$ lb. per sq. in.

22. A uniform beam AB of length 8 ft. is cantilevered at A, the end B being free. The mid-point C is supported at the same horizontal level as A. If the beam carries a uniformly distributed load of 150 lb. per ft., find the external moment that must be applied at C to bring the end B to the same level as A and C. Take EI as equal to 1.5×10^5 lb. in.² (R.Ae.S.)

23. The ordinates of the loading curve for a cantilever wing are as follows :—

Distance from root (ft.) . .	0	4	8	12	16	17	18	19	20
Loading (lb. per ft.) . . .	225	218	198	170	140	130	112	78	0

Assuming that the product EI is constant along the span, and equal to 25×10^7 lb. in.² determine the tip deflection.

24. The support for a bomb consists of a simple beam AC of uniform cross-section. The beam is supported at A, B and C by cross-members which prevent vertical deflection of the beam at these points, but which allow the beam to change its slope freely under load. The distances AB and BC are each 20 in. The bomb is attached at points D and E, the mid-points of AB and BC, and applies loads of 0.6 ton and 0.4 ton respectively at these points. Estimate and plot the bending moment along the beam. (R.Ae.S.)

25. The uniform beam ABCD is simply supported at A, B, C and D. If a load of 500 lb. is applied halfway along AB, and a load of 1000 lb. is applied halfway along CD, both loads being in the same direction determine the bending moment in the beam at B and C. A, B, C and D are all on the same level and $AB = BC = CD = 6$ ft. (R.Ae.S.)

26. ABCD is a uniform beam simply supported at A, B, C, D, the supports being at the same level. $AB = BC = CD = 5$ ft., and the beam carries loads of 200 lb. per ft. over AB and CD, and 400 lb. per ft. over BC. Find what moment must be applied externally at B to keep the beam horizontal at B. Also find what difference this makes to the reaction at C. (R.Ae.S.)

27. A beam of length 40 in. is supported at its ends and at its centre by flexible supports. Midway between each end support and the centre support the beam carries equal concentrated loads of magnitude W . The value of EI of the beam in the plane of bending is 2.5×10^6 lb. in.² The flexibility of the supports is such that a load R applied to the centre support would deflect it $4R/10^5$ in., while a load R applied to either of the end supports would deflect the support $R/10^5$ in. Calculate the reaction in each support. (R.Ae.S.)

28. Calculate the Euler failing load for a tubular strut which is 60 in. long, and whose cross-section is 1.875 in. diameter with walls 0.036 in. thick. $E = 10.2 \times 10^6$ lb. per sq. in.

29. A sample tube, the characteristics of which are given below, is found to fail under a bending moment of 1140 lb. in. A similar tube is subjected to a compressive end load of P lb. offset $\frac{5}{8}$ in. from the tube axis. Determine the maximum value of this load if the tube does not fail. Diameter = 1 in. Length = 15 in. Radius of gyration = 0.341 in. Area = 0.109 sq. in. $E = 10^7$ lb. per sq. in. (R.Ae.S.)

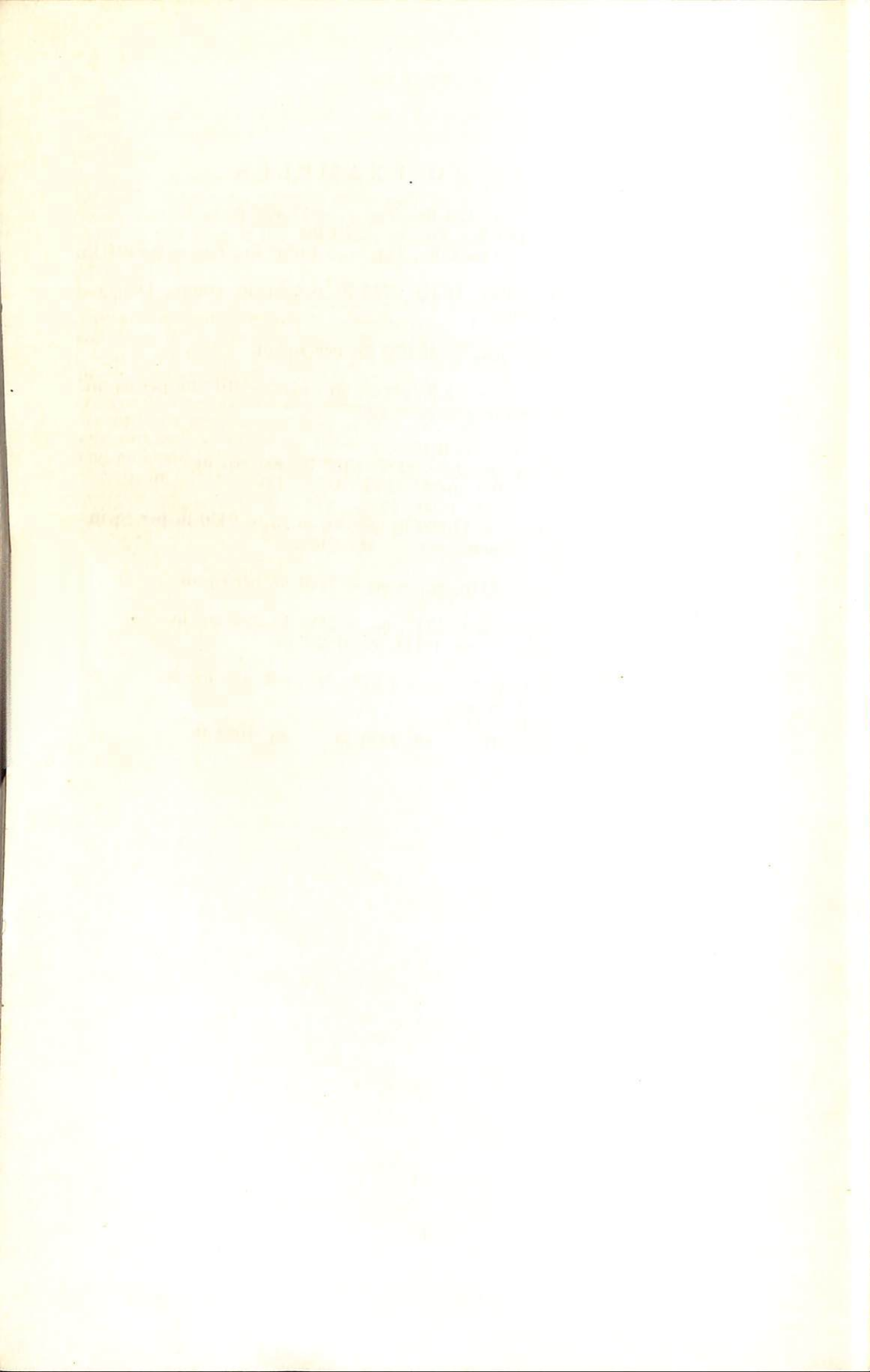
30. A strut carries a compressive end load of 5000 lb., and a concentrated lateral load at the centre of 100 lb. (downwards). If the end moments are 6000 lb. in. and 4000 lb. in., both positive, the second moment of area of the cross-section is 0.5 in.^4 , the length of the strut is 8 ft., and E is 30×10^6 lb. per sq. in., determine the maximum shear and bending moment. (R.Ae.S.)

31. A beam carries a uniformly distributed up load of 10 lb. per in., and an eccentrically applied end load of 6000 lb. The eccentricity of loading is 1 in. at one end and 0.5 in. at the other end, the offset being such as to produce positive end moments. The length of the beam is 80 in., I is 5.8 in.^4 and E is 1.5×10^6 lb. per sq. in. Determine and plot the bending moment along the beam. (R.Ae.S.)

ANSWERS TO EXAMPLES

1. $T_{DA} = 31,500 \text{ lb.}$, $T_{DB} = -13,370 \text{ lb.}$, $T_{DC} = -17,470 \text{ lb.}$
2. $T_{OA} = 260 \text{ lb.}$, $T_{OB} = -140 \text{ lb.}$, $T_{OC} = -310 \text{ lb.}$
3. $M_B = 0.75V \text{ lb. ft.}$, $T_{BE} = -0.667V \text{ lb.}$, $T_{BD} = -1.06V \text{ lb.}$, $T_{BC} = 0.439V \text{ lb.}$
4. 0.366 ton.
5. Vert. 1098 lb. per sq. in. comp., Horiz. 2745 lb. per sq. in. comp., Diagonal 9890 lb. per sq. in. tension.
6. 0.17 in.
7. $f_1 = 88,800 \text{ lb. per sq. in.}$, $q_{\max} = 46,050 \text{ lb. per sq. in.}$
8. 0.13 in.
9. $f_1 = 1621 \text{ lb. per sq. in.}$, $f_2 = -415 \text{ lb. per sq. in.}$, $q_{\max} = 1018 \text{ lb. per sq. in.}$
10. $f_{1,2} = q(\cot \alpha \pm \operatorname{cosec} \alpha)$, where $q = q_1 = q_2$.
11. 4.36°
12. $E = 32 \times 10^6 \text{ lb. per sq. in.}$, $\sigma = 0.2565$,
 $G = 12.76 \times 10^6 \text{ lb. per sq. in.}$, $K = 21.9 \times 10^6 \text{ lb. per sq. in.}$
13. Distance from N.A., in. =

0	0.25	0.50	0.75	1.0	
$q \text{ lb. per sq. in.}$	2122	1980	1592	928	0
14. $q_1 = 11,330 \text{ lb. per sq. in.}$, $q_2 = 12,800 \text{ lb. per sq. in.}$, $q_3 = 9430 \text{ lb. per sq. in.}$
15. 0.3775 in. (using mid-line dimensions). 16. 5.67° .
17. 0.0592 in., 46,800 lb. ft.
18. $q_{ac} = 2355$, $q_{ab} = 547$, $q_{be} = 2410$, $q_{bd} = q_{bf} = 3100 \text{ lb. per sq. in.}$,
 $\theta = 0.04^\circ \text{ per ft.}$
19. $q_{ac} = 448$, $q_{ab} = 2115$, $q_{bd} = q_{bf} = 1317$, $q_{be} = 2885 \text{ lb. per sq. in.}$
20. 6.225 in. behind front spar. 21. 0.215 in., 0.225 in.
22. 2200 lb. ft. 23. 19 in.
24. $M_A = M_C = 0$, $M_D = 0.5625$, $M_B = -1.875$, $M_E = 0.0625 \text{ ton in.}$
25. $M_B = -150$, $M_C = -525 \text{ lb. ft.}$
26. 2950 lb. ft., R_C increased by 393 lb.
27. Centre 1.285W, Outer 0.3075W. 28. 2474 lb. 29. 1088 lb.
30. 6060 lb. in., 74 lb.



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